Journal of Statistical Physics, Vol. 123, No. 1, April 2006 (© 2006)
DOI: 10.1007/s10955-006-9023-0

Bethe Ansatz Solution of Discrete Time Stochastic Processes with Fully Parallel Update

A. M. Povolotsky[1,2] and J. F. F. Mendes[1]

Received November 22, 2004; accepted December 16, 2005
Published Online: March 28, 2006

We present the Bethe ansatz solution for the discrete time zero range and asymmetric exclusion processes with fully parallel dynamics. The model depends on two parameters: p, the probability of single particle hopping, and q, the deformation parameter, which in the general case, $|q| < 1$, is responsible for long range interaction between particles. The particular case $q = 0$ corresponds to the Nagel-Schreckenberg traffic model with $v_{\max} = 1$. As a result, we obtain the largest eigenvalue of the equation for the generating function of the distance travelled by particles. For the case $q = 0$ the result is obtained for arbitrary size of the lattice and number of particles. In the general case we study the model in the scaling limit and obtain the universal form specific for the Kardar-Parisi-Zhang universality class. We describe the phase transition occurring in the limit $p \to 1$ when $q < 0$.

KEY WORDS: Asymmetric exclusion process, Bethe ansatz, cellular automata.

1. INTRODUCTION

One-dimensional models of stochastic processes attracted much attention last decade. Being related to several natural phenomena like the interface growth,[(1)] the traffic flow,[(2)] and the self-organized criticality,[(3)] they admit an exact calculation of many physical quantities, which can not be obtained with mean-field approach. Such models served as a testing ground for the description of many interesting effects specific for nonequilibrium systems like, boundary induced phase transitions,[(4,5)] shock waves[(6)] and condensation transition.[(7)]

The most prominent in the physical community one-dimensional stochastic model is the Asymmetric Simple Exclusion Process (ASEP). The interest in this

[1] Physics Department, University of Aveiro, Campus de Santiago, 3810-193, Aveiro, Portugal.
[2] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia; e-mail: povam@thsun1.jinr.ru

model was inspired by its Bethe ansatz solution,[8,9] which became the first direct exact calculation of the dynamical exponent of Kardar-Parisi-Zhang equation.[1] Since then plenty of exact results were obtained in this direction using both the Bethe ansatz[10] and the matrix product method.[11] The advantage of the former is the possibility of consideration of time dependent quantities rather than only the stationary ones. Among the results obtained for the ASEP with the help of the Bethe Ansatz there are the crossover scaling functions for the Kardar-Parisi-Zhang universality class,[12] the large deviation function of the Kardar-Parisi-Zhang-type interface,[13,14] the time dependent correlation functions on the infinite[15] and the periodic lattices.[16]

Though the Bethe ansatz solvability opens rich opportunities for obtaining exact results, it implies restrictive limitations for the dynamical rules, such that the range of the Bethe ansatz solvable models is not too wide. Besides the ASEP, the examples of the particle hopping models, which admit the Bethe ansatz solution, are the asymmetric diffusion models,[17,18] generalized drop-push models,[19−21] and the asymmetric avalanche process (ASAP).[22] Most of the models of stochastic processes solved by the Bethe ansatz are formulated in terms of continuous time dynamics or random sequential update, which allows one to use the analogy with the integrable quantum chains. The results for the processes with discrete time parallel update are rare.[23] On the other hand in conventional theory of quantum integrable systems the fundamental role is played by the two dimensional vertex models, all the quantum spin chains and continuous quantum models being their particular limiting cases.[24] In the theory of one-dimensional stochastic processes the same role could be played by the models with discrete time parallel update, i.e. so called stochastic cellular automata.[25] Having plenty of real applications, the cellular automata give splendid opportunity for doing large scale numerical simulations. Thus, finding the Bethe ansatz solutions for discrete time models with fully parallel update would be of interest. The lack of such solutions owes probably to more complicated structure of the stationary state of models with the fully parallel update, which makes the application of the Bethe ansatz more subtle.

Recently the Bethe ansatz was applied to solve the continuous time zero-range process (ZRP)[26] with the nonuniform stationary state.[27] The solution was based on the Bethe ansatz weighted with the stationary weights of corresponding configurations. The simple structure of the stationary state of the ZRP allowed finding the one-parametric family of the hopping rates, which provides the Bethe ansatz integrability of the model. In the present article we use the same trick to solve much more general model of the ZRP with fully parallel update, which is defined by two-parametric family of hopping probabilities. As particular limiting cases of the parameters the model includes q-boson asymmetric diffusion model,[18] which in turn includes the drop-push[19] and phase model,[28] and the ASAP.[22] Simple mapping allows one to consider also the ASEP-like model that obeys the exclusion rule. In general it includes the long range interaction between particles, which makes the hopping probabilities dependent on the length of queue of particles next to the hopping one. In the simplest case, when the latter interaction is switched off, the model reduces to the Nagel-Schreckenberg traffic model[(2)] with the maximal velocity $v_{\max} = 1$. In the article we find the eigenfunctions and eigenvalues of the equation for the generating function of the distance travelled by particles and obtain the Bethe equations for both ASEP and ZRP cases. We analyze the solution of the Bethe equations corresponding to the largest eigenvalue of the equation for the generating function of the distance travelled by particles. The analysis shows that a particular limit of parameters of the model reveals the second order phase transition. When the density of particles reaches the critical value, the new phase emerges, which changes the dependence of the average flow of particles on the density of particles. The phase transition turns out to be intimately related to the intermittent-to-continuous flow transition in the ASAP, and the jamming transition in the traffic models. We analyze this transition in detail.

The article is organized as follows. In the Sec. 2 we formulate the model and discuss the basic results of the article. In the Sec. 3 we give the Bethe ansatz solution of the equation for the generating function of the moments of the distance travelled by particles. In the Sec. 4 we study the long time behavior of this equation and obtain its largest eigenvalue. The special form of the Bethe equations in the particular case, which corresponds to the Nagel-Schreckenberg model with $v_{\max} = 1$, allows one to proceed with the calculations for arbitrary size of the lattice and number of particles. In the other cases we obtain the results in the scaling limit. In the Sec. 5 we discuss the particular limit of the model, which exhibits the phase transition. In the Sec. 6 we discuss the calculation of the stationary correlation functions with the partition function formalism. A short summary is given in the Sec. 7.

2. MODEL DEFINITION AND MAIN RESULTS

2.1. Zero Range Process

The system under consideration can be most naturally formulated in terms of the discrete time ZRP. Let us consider the one-dimensional periodic lattice consisting of N sites with M particles on it. Every site can hold any number of particles. A particle from an occupied site jumps to the next site forward with probability $p(n)$, which depends only on the occupation number n of a site of departure. The system evolves step by step in the discrete time t, all sites being updated simultaneously at every step. Thus, as a result of the update the configuration C, defined by the set of occupation numbers

$$C = \{n_1, \ldots, n_N\}, \tag{1}$$

changes as follows

$$\{n_1, \ldots, n_N\} \to \{n_1 - k_2 + k_1, \ldots, n_N - k_1 + k_N\}. \tag{2}$$

Here the variable k_i , taking values 1 or 0, denotes the number of particles arriving at the site i. According to these dynamical rules, all k_i-s associated to the same time step are independent random variables with the distribution, which depends on the occupation number of the site $i - 1$

$$P(k_i = 1) = p(n_{i-1}), \qquad P(k_i = 0) = 1 - p(n_{i-1}). \tag{3}$$

The probability $P_t(C)$ for the system to be in a configuration C at time step t obeys the Markov equation

$$P_{t+1}(C) = \sum_{\{C'\}} T\left(C, C'\right) P_t\left(C'\right), \tag{4}$$

where $T(C, C')$ is the probability of the transition from C' to C. This equation is known to have a unique stationary solution,[7] which belongs to the class of the so called product measures, i.e. the probability of a configuration is given by the product of one-site factors

$$P_{st}(n_1, \ldots, n_N) = \frac{1}{Z(N, M)} \prod_{i=1}^{N} f(n_i), \tag{5}$$

where the one-site factor $f(n)$ is defined as follows

$$f(0) = 1 - p(1),$$
$$f(n) = \frac{1 - p(1)}{1 - p(n)} \prod_{i=1}^{n} \frac{1 - p(i)}{p(i)}, \quad n > 0 \tag{6}$$

and $Z(N, M)$ is the normalization constant. The properties of this stationary measure have been intensively investigated particularly due to the possibility of condensation transition in such systems.[7] The question we are going to address below is: which dynamics leading to such a stationary measure admits the exact solution?

2.2. Associated Exclusion Process

A simple mapping allows one to convert ZRP into ASEP. Given a ZRP configuration, we represent every site occupied with n particles as a sequence of n sites occupied each with one particle plus one empty site ahead (see Fig. 1). Thus, we add M extra sites to the lattice, so that its size becomes $L = M + N$. Obviously the exclusion constraint is imposed now. The evolution started with such a configuration is completely defined by the above ZRP dynamics. Specifically,

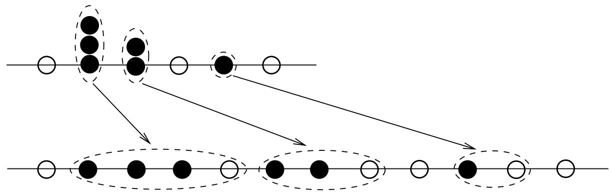

Fig. 1. ZRP-ASEP mapping. Each site occupied with n particles is mapped to the sequence of n sites occupied each by one particle with one empty site ahead.

any particle, which belongs to a cluster of n particles and has empty site ahead (i.e. the first particle of the cluster), jumps forward with the probability $p(n)$ or stays with the probability $(1 - p(n))$. Thus, we obtain the generalized ASEP with the asymmetric long range interaction, which we refer to as the ASEP associated to given ZRP. Note that the above ZRP-ASEP relation applied on the periodic lattice establishes correspondence between the sequences of configurations rather than between the configurations themselves. This fact can be illustrated with a simple example. Consider the following sequence of steps of the ZRP evolution. At the first step a particle jumps from an occupied site i to the next site $i + 1$, at the second from $i + 1$ to $i + 2$ and so on until a particle comes to the site i from $i - 1$ restoring the initial configuration. Reconstruction of the corresponding sequence of the ASEP configurations implies that when a particle joins a cluster of particles from the behind, the front particle of the same cluster takes the next step, which leads to the shift of the cluster one step backward. As a result we finally come to the configuration translated as a whole one step backward with respect to the initial one. This difference does not allow the ZRP and the ASEP to be considered as one model. There are, however, some quantities, which are not sensitive to translations,[3] and therefore are identical for two models. For example the stationary measure of the associated ASEP is given by the expression (5), though $f(n)$ corresponds now to a cluster of n particles with one empty site ahead. Apparently, the probability of any configuration does not change under the translation of the configuration as a whole. Below we consider the generating function of the total distance travelled by particles, Y_t, in the infinite time limit, $lim_{t\to\infty} \ln\langle e^{\gamma Y_t}\rangle/t$,which is also the same for the ZRP and the ASEP associated to it. The only thing we should keep in mind in such cases is that in ZRP the numbers N and M denote the lattice size and the number of particles respectively, while in the associated ASEP N is the number of holes, M is the number of particles, and the size of the lattice is $L = N + M$. Of course, time- or spatially- dependent quantities in general require separate consideration for each model. In this article

[3] The exact meaning of the term "insensitive to translations" will be clarified in the Sec. 3.

we first consider ZRP dynamics and then point out the difference of solution for the of associated ASEP.

In the end of the discussion of the ZRP-ASEP correspondence we should mention that another version of the ASEP can be obtained from the above one by the particle-hole transformation. Then the hopping probabilities will depend on the length of headway in front of the particle like for example the hopping rates in the bus route model.[29] Though the latter formulation seems more naturally related to the real traffic, we will use the former one to keep the direct connection with the ZRP. Of course the results for both versions can be easily related to each other.

2.3. The Results

To formulate basic result of the article we introduce the following generating function

$$F_t(C) = \sum_{Y=0}^{\infty} e^{\gamma Y} P_t(C, Y),$$

where $P_t(C, Y)$ is the joint probability for the system to be in configuration C at time t, the total distance travelled by particles Y_t being equal to Y. By definition the generating function, $F_t(C)$, coincides with the probability $P_t(C)$ of the configuration C in the particular case $\gamma = 0$. The function $F_t(C)$ obeys the evolution equation similar to (4)

$$F_{t+1}(C) = \sum_{\{C'\}} e^{\gamma \mathcal{N}(C,C')} T\left(C, C'\right) F_t\left(C'\right). \tag{7}$$

The term $e^{\gamma \mathcal{N}(C,C')}$ accounts the increase of the total distance Y_t, $\mathcal{N}(C, C')$ being the number of particles, which make a step during the transition from C' to C. Below we show that the eigenfunctions of this equation, satisfying the following eigenfunction problem

$$\Lambda(\gamma) F_{\Lambda}(C) = \sum_{\{C'\}} e^{\gamma \mathcal{N}(C,C')} T\left(C, C'\right) F_{\Lambda}\left(C'\right), \tag{8}$$

have the form of the Bethe ansatz weighted with the weights of stationary configurations provided that the hopping probabilities has the following form

$$p(n) = p \times [n]_q, \quad n = 1, 2, 3, \ldots, \tag{9}$$

where $[n]_q$ are the so called q-numbers

$$[n]_q = \frac{1 - q^n}{1 - q}. \tag{10}$$

Thus, as usual, the Bethe ansatz to be applicable, the infinite in general set of toppling probabilities should be reduced to two parametric family, the parameters being p and q. Formally, there are no any further limitations on the parameters. However, once we want $p(n)$ to be probabilities, they should be less than or equal to one and nonnegative. It is obviously correct for any n if

$$0 \leq p \leq 1 - q \quad \text{and} \quad |q| \leq 1. \tag{11}$$

If $q > 1$, and an integer $n^* > 1$ exists, such that $1/[n^* + 1]_q < p < 1/[n^*]_q$, we can still formulate the model with a finite number of particles $M < n^*$. The only way to consider an arbitrary number of particles with $q > 1$ is to consider the limit $p \to +0$. Particularly, if we put $p = \delta\tau$, where $\delta\tau$ is infinitesimally small, the model turns into the continuous time version of ZRP (or associated ASEP) known also as q-boson asymmetric diffusion model.[17] In this model the Poisson rate $u(n)$ of the hopping of a particle from a site is given by $u(n) = [n]_q$, q taking on values in the range $q \in (-1, \infty)$. Such continuous time limit seems to be the only possibility for q to exceed 1 . This model was studied in.[17,27] The case $(p > 1 - q,\ |q| \leq 1)$ also implies $p(n) > 1$ for some finite $n > n^*$ and thus do not allow consideration of arbitrary M, which is of practical interest. Thus, below we concentrate on the domain (11).

The other particular limits of the model under consideration, which reproduce the models studied before are as follows:

– *The Nagel-Schreckenberg traffic model with* $v_{\max} = 1$ corresponds to the ASEP with $q = 0$, when the probabilities $p(n)$ are independent of n, $p(n) \equiv p$. This case will be considered separately because, due to the special factorized form of the Bethe equations, it can be treated for arbitrary finite N and M.

– *The asymmetric avalanche process* corresponds to the limit

$$(1 - p) \to 0. \tag{12}$$

Originally the ASAP has been formulated as follows. In a stable state, M particles are located on a ring of N sites, each site being occupied at most by one particle. At any moment of continuous time each particle can jump one step forward with Poisson rate 1. If a site i contains more than one particle $n_i > 1$, it becomes unstable, and must relax immediately by spilling forward either n particles with probability μ_n or $n - 1$ particles with probability $1 - \mu_n$. The relaxation stops when all sites become stable again with $n_i \leq 1$ for any i. The period of subsequent relaxation events is called avalanche. The avalanche is implied to be infinitely fast with respect to the continuous time. The toppling probabilities, which ensure the exact solvability of the model can also be written in terms of $q-$numbers

$$\mu_n = 1 - [n]_q, \quad -1 < q \leq 0, \tag{13}$$

Below we show that the ASAP can be obtained as the continuous time limit of the ZRP observed from the moving reference frame, one step of the discrete time t being associated with the infinitesimal time interval $(1-p)$.

Let us now turn to the physical results we can extract from the Bethe ansatz solution. To this end, we note that the large time behavior of the generating function of the distance travelled by particles $\langle e^{\gamma Y_t}\rangle$ is defined by the largest eigenvalue $\Lambda_0(\gamma)$ of the Eq. (8)

$$\left\langle e^{\gamma Y_t}\right\rangle = \sum_{\{C\}} F_t(C) \sim (\Lambda_0(\gamma))^t, \quad t \to \infty.$$

Respectively, the logarithm of the largest eigenvalue $\Lambda_0(\gamma)$ is the generating function of the cumulants of Y_t in the infinite time limit

$$\lim_{t\to\infty} \frac{\left\langle Y_t^n\right\rangle_c}{t} = \left.\frac{\partial^n \ln \Lambda_0(\gamma)}{\partial \gamma^n}\right|_{\gamma=0}.$$

Here the angle brackets denote the cumulants rather than moments of the particle current distribution, which is emphasized with the subscript c. Below we give the results for $\Lambda_0(\gamma)$ given in terms of two parameters, one of which is q introduced above, and the other is λ defined as follows

$$\lambda = \frac{p}{1-p-q}, \tag{14}$$

which captures all the dependence on the above p.

* *For $q = 0$, which corresponds to the Nagel-Schreckenberg traffic model with unit maximal velocity, $v_{\max} = 1$,* we obtain

$$\ln \Lambda_0(\gamma) = -\lambda \sum_{n=1}^{\infty} \frac{B^n}{n} \binom{Ln-2}{Nn-1} {}_2F_1\left(\begin{matrix}1-Mn, 1-nN\\ 2-nL\end{matrix}; -\lambda\right), \tag{15}$$

$$\gamma = -\frac{1}{M} \sum_{n=1}^{\infty} \frac{B^n}{n} \binom{Ln-1}{Nn} {}_2F_1\left(\begin{matrix}-Mn, -nN\\ 1-nL\end{matrix}; -\lambda\right), \tag{16}$$

where $\Lambda_0(\gamma)$ is defined parametrically, both $\Lambda_0(\gamma)$ and γ being power series in the variable B, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is a binomial coefficient, ${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; x\right)$ is the Gauss hypergeometric function, and $L = N + M$.

* *For an arbitrary q we obtain the results in the scaling limit $N \to \infty$, $M \to \infty$, $M/N = \rho = const$, $\gamma N^{3/2} = const$*

$$\ln \Lambda_0(\gamma) = N\phi\gamma + k_1 N^{-3/2} G\left(k_2 N^{3/2}\gamma\right), \tag{17}$$

where $G(x)$ is a universal scaling function defined parametrically, both $G(x)$ and x being the functions of the same parameter $\varkappa$

$$G(x) = -\sum_{s=1}^{\infty} \frac{\varkappa^s}{s^{5/2}} \tag{18}$$

$$x = -\sum_{s=1}^{\infty} \frac{\varkappa^s}{s^{3/2}}. \tag{19}$$

This function appeared before in the studies of the ASEP,[13] ASAP[32] and polymers in random media[30] believed to belong to the KPZ universality class. The parameters ϕ, k_1, k_2 are the model dependent constants, which depend on two parameters λ and q and the density of particles ρ. An explicit form of these constants will be derived in the following sections.

The results in the limit $p \to 1, q < 0$ are of special interest. Below we show that in this limit the model exhibits a phase transition, which is closely related to the intermittent-continuous flow transition in the ASAP and the jamming transition in traffic models. When the density of particles is less than critical density

$$\rho_c = 1/(1-q), \tag{20}$$

almost all particles synchronously jump forward with rare exceptions, which happen with probability of order of $(1-p)$. Thus the average flow of particles is equal to the density of particles with small correction of order of $(1-p)$.

$$\phi\,(\rho < \rho_c) \simeq \rho \tag{21}$$

However, when the density approaches ρ_c the average flow ϕ stops growing.

$$\phi\,(\rho \geq \rho_c) \simeq \rho_c \tag{22}$$

It turns out that a fraction of particles gets stuck at a small fraction of sites instead of being involved into the particle flow. As a result the density of the particles involved into the flow is kept equal to ρ_c . This phenomenon can be treated as a phase separation. The new phase consisting of immobile particles emerges at the critical point. Any increase of the density above the critical point leads to the growth of this phase, while the density of mobile particles remains unchanged. In terms of the associated ASEP or the traffic models one can describe the situation as an emergence of a small number of traffic jams with average length diverging, when $(1-p)$ decreases. Such behaviour leads to singularities in the constants ϕ, k_1, k_2, Eq. (17), which define the large deviations of the particle current from the average. Below we analyze them in detail.

We also study the stationary measure of the model with the canonical partition function formalism. The calculation of the stationary correlation functions are discussed. We observe the relations between the Bethe ansatz solution in the thermodynamic limit and the thermodynamic properties of the stationary state. We make a conjecture that the relations found between the parameters characterizing the large deviations of the particle current and the parameters of the stationary state hold for the general ZRP belonging to the KPZ universality class. Finally we analyze the change of the behaviour of the occupation number distribution at the critical point in the limit $p \to 1$. In addition the evaluation of $Z(N, M)$ reveals interesting relation of the model with the theory of q-series,[31] being based on the use of the most famous relation of this theory, q-binomial theorem.

3. BETHE ANSATZ SOLUTION

Let us consider the equation for the generating function $F_t(C)$ for the ZRP,

$$\begin{aligned} F_{t+1}(n_1, \ldots, n_N) = \sum_{\{k_i\}} \prod_{i=1}^{N} & (e^{\gamma} p(n_i - k_i + k_{i+1}))^{k_{i+1}} \\ & \times (1 - p(n_i - k_i + k_{i+1}))^{1-k_{i+1}} \\ & \times F_t(n_1 - k_1 + k_2, \ldots, n_N - k_N + k_1), \end{aligned} \tag{23}$$

where we imply periodic boundary conditions $N + 1 \equiv 1$. The summation is taken over all possible values of $k_1, \ldots, k_N$, which are the numbers of particles arriving at sites $1, \ldots, N$ respectively. They take on the values of either 0 or 1 if $n_i \neq 0$ and of only 0 otherwise.

Before using the Bethe ansatz for the solution of this equation we should make the following remark. Most of the models studied by the coordinate Bethe ansatz have a common property. That is, a system evolves to the stationary state, where all the particle configurations occur with the same probability. This property can be easily understood from the structure of the Bethe eigenfunction. Indeed, the stationary state is given by the groundstate of the evolution operator, which is the eigenfunction with zero eigenvalue and momentum. Such Bethe function does not depend on particle configuration at all and results in the equiprobable ensemble. Apparently the ZRP under consideration is not the case like this. The way how to reduce the problem to the one with uniform groundstate was proposed in.[27] The main idea is to look for the solution in the form,

$$F_t(C) = P_{st}(C) F_t^0(C). \tag{24}$$

Here $P_{st}(C)$ is the stationary probability defined in (5). It is not difficult to check that $F_t^0(C)$ satisfy the following equation

$$\begin{aligned} F_{t+1}^0(n_1, \ldots, n_N) = \sum_{\{k_i\}} \prod_{i=1}^{N} & (e^{\gamma} p(n_i))^{k_i} (1 - p(n_i))^{1-k_i} \\ & \times F_t^0(n_1 - k_1 + k_2, \ldots, n_N - k_N + k_1). \end{aligned} \tag{25}$$

This form of the equation has two important advantages. The first is that the coefficient before $F_t^0(C')$ under the sum is the product of one site factors. The second is that if we formally define $p(0) = 0$, the values of k_i-s will take on values of 0 and 1 with no respect to the value of n_i. To proceed further we should go to a different way of representation of system configurations. Let us define the configuration by the coordinates of particles written in non-decreasing order

$$C = \{x_1, \dots, x_M\}, \quad x_1 \le x_2 \le \dots \le x_M. \tag{26}$$

Obviously this is nothing but the change of notations, both representations being completely equivalent. Below we refer to them as to the occupation number representation and the coordinate representation respectively. We are going to show that the eigenfunctions of the Eq. (25) written in the coordinate representation can be found in form of the Bethe ansatz. We first consider the cases $M = 1$ and $M = 2$ subsequently generalizing them for an arbitrary number of particles.

3.1. The Case $M = 1$

The eigenfunction problem for the master equation for one particle is simply the one for the asymmetric random walk in discrete time.

$$\Lambda(\gamma) F^0(x) = e^\gamma p F^0(x-1) + (1-p) F^0(x) \tag{27}$$

The eigenfunction is to be looked for in the form

$$F^0(x) = z^{-x}, \tag{28}$$

where z is some complex number. Substituting (28) to (27) we obtain the expression of the eigenvalue

$$\Lambda(\gamma) = e^\gamma pz + (1-p).$$

The periodic boundary conditions

$$F^0(x+N) = F^0(x)$$

imply the limitations on the parameter z

$$z^N = 1.$$

3.2. The Case $M = 2$

In this case we consider two particles with coordinates x_1 and x_2. If $x_1 \neq x_2$ the equation is that for the asymmetric random walk for two non-interacting particles

$$\begin{aligned}\Lambda(\gamma)F^0(x_1,x_2) = {} & (e^\gamma p)^2 F^0(x_1-1,x_2-1) \\ & + e^\gamma p(1-p)(F^0(x_1-1,x_2) + F^0(x_1,x_2-1)) \\ & + (1-p)^2 F^0(x_1,x_2),\end{aligned} \tag{29}$$

while the case $x_1 = x_2 = x$ should be treated separately.

$$\Lambda(\gamma) F^0(x, x) = e^\gamma p(2) F^0(x-1, x) + (1 - p(2)) F^0(x, x) \qquad (30)$$

The general strategy of the Bethe ansatz solution is as follows. We want to limit ourselves by the only functional form of the equation for $F^0(C)$ of the form (29). To get rid of the additional constraints imposed by interaction, like that in (30), we recall that the function $F^0(x_1, \ldots, x_M)$, is defined in the domain $x_1 \le x_2 \le \cdots \le x_M$. If we formally set $x_1 = x_2 = x$ in the equation (29), one of the terms in r.h.s., $F^0(x, x-1)$, will be outside of this domain. We could redefine it in such a way, that the Eq. (30) would be satisfied. This leads us to the following constraint.

$$aF^0(x, x) + bF^0(x, x-1) + cF^0(x-1, x) + dF^0(x-1, x-1) = 0, \quad (31)$$

where

$$\begin{aligned} a &= ((1-p)^2 - (1 - p(2))), \\ b &= e^\gamma p(1-p), \\ c &= e^\gamma (p(1-p) - p(2)), \\ d &= (e^\gamma p)^2 \end{aligned}$$

The solution of the Eq. (29) can be looked for in the form of the Bethe ansatz,

$$F^0(x_1, x_2) = A_{1,2} z_1^{-x_1} z_2^{-x_2} + A_{2,1} z_1^{-x_2} z_2^{-x_1}. \qquad (32)$$

Substituting it into Eq. (29) we obtain the expression for the eigenvalue

$$\Lambda(\gamma) = (e^\gamma p z_1 + (1-p))(e^\gamma p z_2 + (1-p)),$$

while the constraint (31) results in the relation between the amplitudes $A_{1,2}$ and $A_{2,1}$.

$$\frac{A_{1,2}}{A_{2,1}} = -\frac{a + bz_1 + cz_2 + dz_1 z_2}{a + bz_2 + cz_1 + dz_1 z_2}.$$

The last step, which makes the scheme self-consistent, is to impose periodic boundary conditions

$$F^0(x_1, x_2) = F^0(x_2, x_1 + N),$$

which lead us to the system of two algebraic equations. The first one is

$$z_1^{-N} = -\frac{a + bz_1 + cz_2 + dz_1 z_2}{a + bz_2 + cz_1 + dz_1 z_2}$$

and the second is obtained by the change $z_1 \longleftrightarrow z_2$.

3.3. The Case of Arbitrary M

For an arbitrary number of particles we will follow the same strategy. We consider the equation for noninteracting particles, which jump forward with the probability p

$$\Lambda(\gamma) F^0(x_1, \ldots, x_M) = \sum_{\{k_i\}} \prod_{i=1}^{M} (e^\gamma p)^{k_i} (1-p)^{1-k_i} \times F^0(x_1 - k_1, \ldots, x_M - k_M). \tag{33}$$

Here, all the numbers k_i-s run over values 1 and 0. Our aim is to redefine the terms beyond the physical domain, (26), in terms of ones within this domain in order to satisfy the equation with ZRP interaction (25) written in the coordinate representation. This redefinition results in many constraints on the terms $F^0(\ldots)$. As it will be seen below, the Bethe ansatz to be applicable all these constraints should be reducible to the only one,

$$0 = aF^0(\ldots, x, x, \ldots) + bF^0(\ldots, x, x-1, \ldots) + cF^0(\ldots, x-1, x \ldots) + dF^0(\ldots, x-1, x-1, \ldots), \tag{34}$$

which has been found for the two particle case. To this end, we first recall that the r.h.s. of Eq. (25) is the sum of the terms $F^0(\ldots)$ corresponding to configurations, from which the system can come to the configuration in l.h.s. with coefficients factorized into the product of one-site terms. Therefore, the processes corresponding to a particle arriving at a site can be considered for each site separately. Consider, for instance, Eq. (25), where the argument of the function in l.h.s. is a configuration with a site x occupied by n particles.

$$\Lambda(\gamma)F^0(\ldots, (x)^n, \ldots) = \sum\nolimits'_{\{k_i\}} \prod\nolimits'_{i \neq x} (e^\gamma p(n_i))^{k_i} (1 - p(n_i))^{1-k_i} \times [e^\gamma p(n) F^0(\ldots, (x-1), (x)^{n-1}, \ldots) + (1 - p(n)) F^0(\ldots, (x)^n, \ldots)] \tag{35}$$

The terms of r.h.s. can be grouped in pairs shown in square brackets, which correspond to the processes, in which a particle comes or does not come to the site x. Here, $(x)^n$ denotes n successive arguments equal to x, i.e. the site x is occupied by n particles. The primed summation and product run over all sites apart of x. Obviously the coefficients in the equations for noninteracting particles can be factorized in similar way, such that the term corresponding to the site x looks as follows

$$\sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} (e^{\gamma} p)^{\sum_{i=1}^{n} k_i} (1-p)^{n-\sum_{i=1}^{n} k_i} \\ \times F^0(\ldots, x-k_1, \ldots, x-k_n, \ldots,). \tag{36}$$

We want this term to be equal to the one in the square brackets of Eq. (35). If $n = 2$ the form of the term in square brackets is similar to the r.h.s of Eq. (30) and can be treated (i.e. reduced to the noninteracting form (35) using the constraint (34). The equality for general n can be proved by induction. Suppose it is valid for $n-1$. Then we can perform the summations in Eq. (35) over k_i-s for $i = 2, \ldots, n$, resulting in

$$\begin{aligned}(1-p)[e^{\gamma} p(n-1) F^0(\ldots, x, (x-1), (x)^{n-2}, \ldots) \\ +(1-p(n-1)) F^0(\ldots, (x)^n, \ldots)] \\ +e^{\gamma} p[e^{\gamma} p(n-1) F^0(\ldots, (x-1)^2, (x)^{n-2}, \ldots) \\ +(1-p(n-1)) F^0(\ldots, (x-1), (x)^{n-1}, \ldots)].\end{aligned} \tag{37}$$

The first summand contains the term $F^0(\ldots, x, (x-1), (x)^{n-2}, \ldots)$, which, being beyond the physical region (26), can be expressed in terms of ones inside the physical region by using the constraint (34). Equating the coefficients of the terms $F^0(\ldots)$ with the same arguments of Eqs. (36) and (37) we obtain the relation between $p(n)$ and $p(n-1)$

$$p(n) = p + qp(n-1), \tag{38}$$

where we introduce the notation q defined as follows

$$p(2) \equiv p \times (1+q). \tag{39}$$

The recurrent relation (38) can be solved in terms of $q-$numbers as was claimed in Eqs. (9,10).

Thus, we have shown that in the physical domain (26) the solution of the equation for ZRP coincides with the solution of the equation for non-interacting particles if the latter satisfies the constraint (34). Then we can use the Bethe ansatz

$$F^0(x_1, \ldots, x_M) = \sum_{\{\sigma_1, \ldots, \sigma_M\}} A_{\sigma_1, \ldots, \sigma_M} \prod_{i=1}^{M} z_{\sigma_i}^{-x_i} \tag{40}$$

for the eigenfunction of the free equation. Here $z_1, \ldots, z_M$ are complex numbers, the summation is taken over all $p!$ permutations $\{\sigma_1, \ldots, \sigma_M\}$ of the numbers $(1, \ldots, M)$. Substituting the Bethe ansatz (40) into the Eq. (33), we obtain the expression for the eigenvalue

$$\Lambda(\gamma) = \prod_{i=1}^{M} \left(e^{\gamma} p z_i + (1-p)\right). \tag{41}$$

Substituting it to the constraint (34) we obtain the relation between pairs of the amplitudes $A_{\sigma_1,\dots,\sigma_M}$, which differ from each other only in two indices permuted

$$\frac{A_{\dots j,i \dots}}{A_{\dots i,j \dots}} = S_{i,j} \equiv -\frac{a + bz_j + cz_i + dz_i z_j}{a + bz_i + cz_j + dz_i zj}. \tag{42}$$

With the help of this relation one can obtain all the amplitudes $A_{\sigma_1,\dots,\sigma_M}$ in terms of only one, say $A_{1,\cdots,M}$, by successive permutations of indices, which results in multiplication by the factors S_{ij}. For example, for three particle case we have

$$A_{2,1,3} = S_{1,2}A_{1,2,3}, \quad A_{2,3,1} = S_{1,3}S_{1,2}A_{1,2,3}, \quad A_{3,1,2} = S_{2,3}S_{1,3}A_{1,2,3}$$
$$A_{1,3,2} = S_{2,3}A_{1,2,3}, \quad A_{3,2,1} = S_{2,3}S_{1,3}S_{1,2}A_{1,2,3}. \tag{43}$$

In general, the procedure to be uniquely defined, it should be consistent with the structure of the permutation group. Specifically, if $\widehat{\sigma_i}$ is an elementary transposition that permutes the numbers at *i*-th and $(i+1)$-th positions, it satisfies the following relations.

$$\widehat{\sigma_i}\widehat{\sigma_{i+1}}\widehat{\sigma_i} = \widehat{\sigma_{i+1}}\widehat{\sigma_i}\widehat{\sigma_{i+1}}$$
$$\widehat{\sigma_i}^2 = 1 \tag{44}$$

If the numbers at positions i, $(i+1)$, and $(i+2)$ are j, k, and l respectively, two relations of Eq. (44) are reduced to

$$S_{j,k}S_{j,l}S_{k,l} = S_{k,l}S_{j,l}S_{j,k},$$
$$S_{j,k}S_{k,j} = 1, \tag{45}$$

which are apparently true.

The last step we need to do is to impose the periodic boundary conditions.

$$F^0(x_{1,}, \dots, x_M) = F^0(x_2, , \dots, x_M, x_1 + N) \tag{46}$$

They are equivalent to the following relations for the amplitudes.

$$A_{\sigma_1,\dots,\sigma_M} = A_{\sigma_2,\dots,\sigma_M,\sigma_1} z_{\sigma_1}^{-N}$$

Consistency of this relation with Eq. (34) results in the system of M algebraic Bethe equations,

$$z_i^{-N} = (-)^{M-1} \prod_{i=1}^{M} \frac{a + bz_i + cz_j + dz_i z_j}{a + bz_j + cz_i + dz_i z_j}. \tag{47}$$

We should emphasize that the crucial step for the Bethe ansatz solvability is the proof that all many-particle interactions can be reduced to the two-particle constraint, Eq. (34). Existence of any other constraints on the eigenfunctions would result in new equations for parameters z_i, which would make the resulting system of algebraic equations overdetermined.

3.4. Bethe Ansatz for the ASEP

As it was noted above, the Bethe ansatz solution for the ASEP is quite similar to that for the ZRP and can be done in parallel. Indeed, if we define the function $F_t^0(C)$ in the same way as it was defined for ZRP, the equations for it can be obtained from those for ZRP by the variable change

$$\{x_{1,}x_2,\dots,x_M\} \to \{x_1, x_2+1,\dots,x_M+M-1\}, \tag{48}$$

which corresponds to the ZRP-ASEP transformation described above. Thus, the free equation does not change its form, with the only difference that the physical domain for the ASEP implies every site to be occupied at most by one particle.

$$x_1 < x_2 < \cdots < x_M. \tag{49}$$

Therefore the Bethe ansatz (40) substituted to the free equation results in the same form of the eigenvalue (41). The two particle constraint, which is used now to redefine the nonphysical terms containing the pair (x, x), have the following form

$$0 = aF^0(x, x+1) + bF^0(x, x) + cF^0(x-1, x+1) + dF^0(x-1, x). \tag{50}$$

If we insert here the Bethe ansatz (40), we obtain the following relation for the amplitudes

$$\frac{A_{1,2}}{A_{2,1}} = -\frac{az_1^{-1} + b + cz_2z_1^{-1} + dz_2}{az_2^{-1} + b + cz_1z_2^{-1} + dz_1}. \tag{51}$$

All the other arguments, which extend the problem to the general $M-$ particle, case are completely the same as those for the ZRP. Finally, the periodic boundary conditions lead to the Bethe equations

$$z_i^{-L} = (-)^{M-1}\prod_{j=1}^{M}\frac{az_i^{-1} + b + cz_jz_i^{-1} + dz_j}{az_j^{-1} + b + cz_iz_j^{-1} + dz_i}. \tag{52}$$

Here we recall that the lattice size is equal now to $L = N + M$ rather than N . This equations can be rewritten in the following form

$$z_i^{-N} = T \times (-)^{M-1}\prod_{j=1}^{M}\frac{a + bz_i + cz_j + dz_iz_j}{a + bz_j + cz_i + dz_iz_j}. \tag{53}$$

where we introduce the notation

$$T = \prod_{j=1}^{M} z_j. \tag{54}$$

This term is the only difference between the Bethe equations for the ZRP and the ASEP. Taking a product of all the equations we obtain,

$$T^L = 1. \tag{55}$$

The term T has a very simple physical meaning. This is the factor that corresponding eigenfunction multiplies by under the unit translation. Thus, the eigenfunction that corresponds to the set $\{z_i\}$ satisfying a relation $T = 1$ is invariant with respect to any translations for both ZRP and ASEP. This is what we meant mentioning the translational invariance above. Apparently the solutions of the Bethe equations, which satisfy the relation $T = 1$ coincide for the ZRP and the ASEP and so do any quantities constructed with them. Below we consider such quantity and, therefore, do not make a distinction between the ZRP and the ASEP.

4. THE LARGEST EIGENVALUE

To proceed further, we introduce new variables y_i defined as follows

$$z_i = e^{-\gamma} \frac{1 - y_i}{1 + \lambda y_i}, \tag{56}$$

where λ is defined in Eq. (14). In these variables the Bethe Eqs. (47) and the eigenvalue (41) simplify to the following form

$$\left(\frac{1 - y_i}{1 + \lambda y_i}\right)^{-N} e^{\gamma N} = (-)^{M-1} \prod_{j=1}^{M} \frac{y_i - q y_j}{y_j - q y_i}, \tag{57}$$

$$\Lambda(\gamma) = \prod_{i=1}^{M} \frac{1 + \lambda q y_i}{1 + \lambda y_i} \tag{58}$$

Let us now consider the eigenstate, which corresponds to the maximal eigenvalue $\Lambda_0(\gamma)$. Note that in the limit $\gamma \to 0$ the Eq. (7) for $F_t(C)$ turns into the Markov equation for the probability (4). The largest eigenvalue of the Markov equation is equal to 1. The corresponding eigenstate is the stationary state (5). It then follows from the definition, that $F_t^0(C)$ in this limit becomes constant, i.e. it is the same for all configurations, and the corresponding solution of the Bethe equations is $z_i = 1$ or $y_i = 0$. If γ deviates from 0 the analyticity and no-crossing of the eigenvalue is guaranteed by Perron-Frobenius theorem. By continuity we also conclude that the parameter T defined above is equal to 1 for arbitrary γ,

$$\prod_{j=1}^{M} e^{-\gamma} \frac{1-y_j}{1+\lambda y_j} = 1. \tag{59}$$

4.1. The Case $q = 0$

In the case $q = 0$, the form of Eq. (57) allows one to obtain the largest eigenvalue for arbitrary M and N.[13] If we define the parameter,

$$B = (-)^{M-1}\, e^{\gamma N} \prod_{j=1}^{M} y_j, \tag{60}$$

the solution of the Bethe equations, which corresponds to the largest eigenvalue, will be given by M roots of the polynomial

$$P(y) = (1+\lambda y)^N\, B - (1-y)^N\, y^M, \tag{61}$$

which approach zero when B tends to zero. Then the sum of a function $f(x)$ analytic in some vicinity of zero over the Bethe roots, can be calculated as the following integral over the contour closed around the roots:

$$\sum_{j=1}^{M} f(y_j) = \oint \frac{dy}{2\pi i} \frac{P'(y)}{P(y)} f(y). \tag{62}$$

Particularly, after the integration by parts the expression for $\ln \Lambda_0(\gamma)$ has the following form:

$$\ln \Lambda_0(\gamma) = \oint \frac{dy}{2\pi i} \frac{\lambda}{1+\lambda y} \ln\left(1 - \frac{B\,(1+\lambda y)^N}{(1-y)^N\, y^M}\right). \tag{63}$$

At the same time the expression for γ as a function of B can be obtained by taking a logarithm of Eq. (59)

$$\gamma = \frac{1}{M} \oint \frac{dy}{2\pi i} \left(\frac{1}{1-y} + \frac{\lambda}{1+\lambda y}\right) \ln\left(1 - \frac{B\,(1+\lambda y)^N}{(1-y)^N\, y^M}\right). \tag{64}$$

To evaluate it we make a series expansion of the logarithms in powers of B and integrate the resulting series term by term. The procedure is valid until the resulting series are convergent. The resulting series, obtained after some algebra by using standard identities for hypergeometric functions, are given in Eqs. (15,16). The large time asymptotics of the cumulants of the distance travelled by particles $\lim_{t\to\infty}\langle Y_t^n\rangle_c/t$ can be obtained by eliminating the parameter B between two series. For instance, the exact value of the average flow of particles is of the following form:

$$\phi = \frac{1}{N}\lim_{t\to\infty}\frac{\langle Y_t\rangle_c}{t} = \frac{M\lambda}{L-1}\frac{{}_2F_1\left(\begin{matrix}1-M, 1-N\\ 2-L\end{matrix}; -\lambda\right)}{{}_2F_1\left(\begin{matrix}-M, -N\\ 1-L\end{matrix}; -\lambda\right)}. \tag{65}$$

In the thermodynamic limit $N \to \infty$, $M \to \infty$, $M/L = c$ the average velocity of particles $\overline{v} = N\phi/M$ saturates to the formula obtained in,[2]

$$\bar{v} = \frac{N}{M}\phi = \frac{1-\sqrt{1-4pc(1-c)}}{2c}. \tag{66}$$

It is straightforward to obtain the other cumulants up to any arbitrary order. It is not difficult also to study the asymptotic behavior of the series in the large N limit. To this end one can evaluate the saddle point asymptotics of the integrals instead of their exact values or use the asymptotic formulas for the binomial coefficients and the hypergeometric functions. The thermodynamic limit however allows significant simplification already at the stage of writing of the Bethe equations, which makes possible to consider more general case of arbitrary value of q.

4.2. The Case of Arbitrary q

The technique used in this section was first developed to study the non-factorizable Bethe equations for the partially ASEP[2,14] and then applied to the study of the ASAP.[32] Since the analysis is quite similar, we outline only the main points of the solution. Technical details can be found in the original papers.

The scheme consists of the following steps. We assume that in the thermodynamic limit, $N \to \infty$, $M \to \infty$, $M/N = \rho$, the roots of the Bethe Eqs. (57) are distributed in the complex plain along some continuous contour Γ with the analytical density $R(x)$, such that the sum of values of an analytic function $\mathrm{f}(x)$ at the roots is given by

$$\sum_{i=1}^{p} \mathrm{f}(x_i) = N\int_\Gamma \mathrm{f}(x)R(x)\,dx. \tag{67}$$

After taking the logarithm and replacing the sum by the integral, the system of Eqs. (57) can be reformulated in terms of a single integral equation for the density.

$$-\ln\left(\frac{1-x}{1+\lambda x}\right) + \gamma - \int_\Gamma \ln\left(\frac{x-qy}{y-qx}\right) R(y)\,dy = i\pi\rho - 2\pi i\int_{x_0}^{x} R\,(x) \tag{68}$$

The r.h.s. comes from the choice of branches of the logarithm, which specifies particular solution corresponding to the largest eigenvalue. The choice coincides with that appeared in the solution of the asymmetric six-vertex model at the conical point[33] and later of the ASEP.[12] It can be also justified by considering the limit $q = 0$, where the locus of the roots can be found explicitly. The point x_0 corresponding to the starting point of roots counting can be any point of the contour. The integral equation should be solved for a particular form of the contour, which is not known *a priory*, and being first assumed should be self-consistently checked after the solution has been obtained. In practice, however, an analytic solution is possible in the very limited number of cases. Fortunately, we can proceed by analogy with the solution of the asymmetric six-vertex model at the conical point[33] choosing the contour closed around zero. The solution of the Eq. (68) yields the density

$$R^{(0)}(x) = \frac{1}{2\pi i x}(\rho - g_{q,\lambda}(x)), \tag{69}$$

where the function $g_{q,\lambda}(x)$ is defined as follows

$$g_{q,\lambda}(x) = \sum_{n=1}^{\infty} \frac{1-(-\lambda)^n}{1-q^n} x^n. \tag{70}$$

This case corresponds to $\gamma = 0$ and hence $\Lambda_0(\gamma) = 1$. Since $R^{(0)}(x)$ is analytic in the ring $0 < |x| < \lambda$, the integration of it along any contour closed in this ring does not depend on its form. Therefore, to fix the form of the contour additional constraints are necessary. Such a constraint was found by Bukman and Shore[33] as follows. They assumed that when γ deviates from zero, the contour becomes discontinuous at some point x_c. It is possible then to solve Eq. (68) perturbatively considering the length of the gap of the contour as a small parameter. It turns out that the solution exists only if the break point x_c satisfies the equation

$$R^{(0)}(x_c) = 0, \tag{71}$$

which fixes the location of x_c as well as the location of all the contour Γ in the limit $\gamma \to 0$. This method however allows a calculation of only the first derivative of $\Lambda_0(\gamma)$ at $\gamma = 0$ in the thermodynamic limit, i.e. the leading asymptotics of the average flow of particles. To study the behaviour of $\Lambda_0(\gamma)$ for nonzero values of γ, a calculation of the finite size corrections to the above expression of $R^{(0)}(x)$ is necessary. To this end, we use the method developed in.[12,13] In fact, once we have the expression for $R^{(0)}(x)$, Eq. (69), it can be directly substituted to the formula for the finite size asymptotic expansion of $R(x)$ obtained in[32] for the ASAP. As a result we obtain the following parametric dependence of $R(x)$ on γ, both being represented as the functions of the same parameter $\varkappa$,

$$R_s = R_s^{(0)} + \frac{1}{N^{3/2}} \frac{1}{2\pi i} \frac{q^{|s|}}{1-q^{|s|}} \times \sum_{n=0}^{\infty} \left(\frac{i}{2N}\right)^n \frac{\Gamma\left(n+\frac{3}{2}\right)}{\pi^{n+\frac{3}{2}}} \frac{c_{2n+1,s}}{\sqrt{2i}} \mathrm{Li}_{n+\frac{3}{2}}(\varkappa) \tag{72}$$

$$\gamma = -\frac{1}{N^{3/2}} \sum_{n=0}^{\infty} \left(\frac{i}{2N}\right)^n \frac{\Gamma\left(n+\frac{3}{2}\right)}{\pi^{n+\frac{3}{2}}} \frac{\bar{c}_{2n+1}}{\sqrt{2i}} \mathrm{Li}_{n+\frac{3}{2}}(\varkappa) \tag{73}$$

Here $\mathrm{Li}_\alpha(x)$ is the polylogarithm function,

$$\mathrm{Li}_\alpha(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^\alpha},$$

R_s and $R_s^{(0)}$ are the Laurent coefficients of $R(x)$ and $R^{(0)}(x)$ respectively defined as follows

$$R(x) = \sum_{s=-\infty}^{\infty} R_s / x^{s+1}, \tag{74}$$

and $c_{n,s}$ and $\bar{c}_n$ are the coefficients of x^n in $(\sum_{k=0}^{\infty} a_k x^k)^s$ and $\log(\sum_{k=0}^{\infty} a_k x^k)$ respectively, where a_n are the coefficients of the inverse expansion $Z^{-1}(x)$ of the function $Z(x) = -\int_{x_0}^{x} R(x)dx$ near the point $Z(x_c)$. As the derivative of $Z(x)$ vanishes at x_c, Eq. (71), the inverse expansion is in powers of $\sqrt{x - Z(x_c)}$,

$$Z^{-1}(x) = \sum_{n=0}^{\infty} a_n \left(x - Z(x_c)\right)^{n/2}.$$

Being substituted to the Abel-Plana formula,[12] which is used to evaluate the difference between the integral in Eq. (68) and the sum in the original Bethe equations, it becomes a source of $1/\sqrt{N}$ corrections to $R^{(0)}(x)$. The location of x_c is to be self-consistently defined from the equation $R(x_c) = 0$. For the first three orders of $1/\sqrt{N}$ expansion the coefficients a_n can be obtained from the inverse expansion of zero order function $Z^{(0)}(x) = -\int_{x_0}^{x} R^{(0)}(x)dx$, while $R^{(0)}(x)$ has been obtained above. The details of calculations can be found in.[32] The sum over the roots of a function $\mathrm{f}(x)$ analytic inside the contour can be evaluated in terms of the Laurent coefficients R_s of $R(x)$

$$\sum_{i=1}^{M} \mathrm{f}(x_i) = 2\pi i N \sum_{s=1}^{\infty} q^{-s} \mathrm{f}_s R_s. \tag{75}$$

and the of Taylor coefficients of $\mathrm{f}(x)$

$$\mathrm{f}(x) = \sum_{s=1}^{\infty} \mathrm{f}_s x^s. \tag{76}$$

This allows evaluation of $\ln \Lambda_0(\gamma)$. Finally the point x_c enters into all the results through the coefficients $c_{2n+1,s}$ and $\overline{c}_{2n+1}$. The final expression for $\Lambda_0(\gamma)$ obtained in the scaling limit $\gamma N^{3/2} = const$ has the form (17–19), which was obtained earlier for the ASEP and the ASAP and claimed to be universal for the KPZ universality class.[1,34] The constants ϕ, k_1, k_2 specific for the model under consideration,

$$\phi = \frac{\lambda x_c}{1+\lambda x_c}, \tag{77}$$

$$k_1 = \sqrt{\frac{x_c}{8\pi}} \frac{2\lambda^2 g'_{q,\lambda}(x_c) + (1+\lambda x_c)\lambda g''_{q,\lambda}(x_c)}{(1+\lambda x_c)^3 (g'_{q,\lambda}(x_c))^{5/2}}, \tag{78}$$

$$k_2 = \sqrt{2\pi x_c g'_{q,\lambda}(x_c)}, \tag{79}$$

are expressed through the derivatives of the function $g_{q,\lambda}(x)$, (Eq. (70)). As it follows from the explicit form of Eq. (71), the unphysical parameter x_c is related to the density ρ by the equation,

$$\rho = g_{q,\lambda}(x_c). \tag{80}$$

In Sec. 6 we show that relations (77,80) can be obtained from the partition function formalism for the stationary state without going into the Bethe ansatz solution. The parameter x_c then turns out to be related to the fugacity of a particle in the infinite system. Thus, this result follows from the properties of the stationary state only, rather than from the solution of the dynamical problem. In general the partition function formalism allows the calculation of any stationary spatial correlations. The Bethe ansatz, however, allows one to probe into really dynamical characteristics, i.e. those, which are related to unequal time correlation functions, such as higher cumulants of the distance travelled by particles. They can be expressed through the constants k_1, k_2 from Eqs. (78,79) by taking derivatives of the formula (15), which yields the following behavior, specific for the KPZ universality class,

$$\langle Y_t^n \rangle_c \sim N^{3(n-1)/2} k_1 k_2^n.$$

Studying the dependence of the results on the physical parameters p and q one should solve the Eq. (80) to find the behavior of x_c. Even without the explicit solution we can say that for generic values of p and q the point x_c is located at the positive part of the real axis in the segment [0, 1), where the function $g_{q,\lambda}(x)$ increases monotonously from 0 to infinity. Apparently the constants ϕ, k_1, k_2 do not have any singularities for x_c varying in this region and, thus, no abrupt change of the behavior is expected. Particularly, the results obtained reproduce the corresponding results for several models studied before. For example, in the continuous time limit, $p \to 0$, keeping only the first order in p we obtain the corresponding constants for $q-$boson asymmetric diffusion model,[17,27] which itself contains the drop push model[19] and phase model[28] as particular cases, and can be related also to the totally ASEP. The only interesting exception is the limiting case $p \to 1$, which turns out to exhibit a kind of phase transition phenomena.

5. ASYMMETRIC AVALANCHE PROCESS AND PHASE SEPARATION IN THE DETERMINISTIC LIMIT

Let us first consider qualitatively what happens in the limit

$$p = 1 - \delta\tau, \delta\tau \to 0, \tag{81}$$

in the domain

$$-1 < q < 0, \quad \rho < 1. \tag{82}$$

In this limit a particle from a single particle (SP) site, i.e. from a site occupied by one particle only, almost definitely takes a step forward at every time step. At the same time, one particle from a many particle (MP) site, i.e. from a site occupied by more than one particle, takes a step with the probability in general less than one. When the jumps of particles from SP sites are purely deterministic, i.e. the equality $p = 1$ holds exactly, the ZRP dynamics becomes "frozen" as soon as the system arrives at any configuration consisting of SP sites only. By "frozen" we mean that at every time step particles from all sites synchronously jump one step forward. Thus, the structure of the configuration remains unchanged up to a uniform translation. Therefore such configurations play the role of absorbing states when $p = 1$. When p is less than 1 by a small value $\delta\tau$, the system can go from one absorbing state to another with the probability of order of $\delta\tau$. This happens if at least one particle decides not to jump.

It is easy to see that the limit under consideration is directly related to the ASAP. Let us look at the system from the comoving reference frame, which moves a step forward at every time step. In this frame particles from the SP sites either stay with the probability $(1 - \delta\tau)$ or take a step backward with the probability $\delta\tau$. At the same time the MP sites play a role of columns (or avalanches) of particles moving backward. Individual history of every such a column develops according to the following rule. If $n > 1$ particles occupy a site, either all n particles jump to the next site backward with probability $\mu_n = (1 - p(n))$ or $(n - 1)$ particles jump and one stays with probability $(1 - \mu_n)$. One can see that the definition of probabilities μ_n coincides with those for the ASAP, Eq. (13), in the leading order in $\delta\tau$. If we associate one step of the discrete time t with the continuous time interval $\delta\tau$, neglect the terms smaller than $\delta\tau$ in the master equation and take the limit $\delta\tau \to 0$, we come to the continuous time master equation for the ASAP.[22] Note, that in the definition of the ASAP at most one avalanche is allowed to exist at a time. In principle, the discrete time dynamics allows for more than one simultaneous MP sites. However, the probability of formation of two or more of them for a finite number of steps is of order of $(\delta\tau)^2$ and, as such, corresponding processes vanish when the limit $\delta\tau \to 0$ is taken.

As shown in[22] the ASAP exhibits the transition from the intermittent to continuous flow regime. In this transition the average avalanche size (the number of particle jumps in an avalanche) diverges in the thermodynamic limit when the density of particles approaches the critical point from below. To explain what this behaviour means in terms of the discrete time model under consideration, we note that the average avalanche size is roughly equal to the average number of particles occupying an MP site times the average number of steps, which an MP site persists for. Let us consider a configuration with one MP site on the lattice. In average, the flow of particles into the MP site is equal to the density of mobile (solitary) particles. The flow out of the MP site with n particles is $p(n)$, which saturates exponentially rapidly to the limiting value $p(\infty) = 1/(1-q)$ when n grows. Therefore, if $\rho < \rho_c \equiv 1/(1-q)$, the outflow exceeds the inflow and the MP site tends to decay for a few steps. As a result the average occupation number and the life time of an MP site below the critical density are finite. On the contrary, if $\rho > \rho_c$, the outflow is less than inflow and the dynamics favours the growth of the MP sites. On the finite lattice the MP site grows at the expense of the density of solitary particles until it absorbs enough particles to equalize inflow and outflow. This happens when the MP site contains $N(\rho - \rho_c)$ particles. Any deviation from this number destroys the balance between the inflow and outflow and the system tends to restore this value again. As a consequence the life time of such an MP site grows exponentially with the system size.[35]

Of course the existence of only one MP site on the lattice is specific for the continuous time ASAP, where the probability of appearance of two MP sites simultaneously is neglible. In the discrete time case a possibility to have many MP sites at the lattice should be considered as well. However, if the characteristic occupation number of MP sites is such a large that the outflow from MP sites can be roughly replaced by $p(\infty)$, the above claim based on the balance between inflow and outflow remains valid: the total number of particles in all MP sites is equal to $N(\rho - \rho_c)$. As a result, the particle current growing linearly with the density below the critical point, stops growing at ρ_c due to appearance of MP sites, the occupation numbers of which grow instead. Of course these arguments describe the limiting situation $p \to 1$. When $\delta\tau$ is finite, the transition is smoothened. As shown in Fig. (2), the flow-density plot, obtained by substituting the result of numerical solution of Eq. (80) to Eq. (77), approaches the form of broken line consisting of two straight segments: $\phi = \rho$ for $\rho < \rho_c$ and $\phi = \rho_c$ for $\rho \geq \rho_c$.

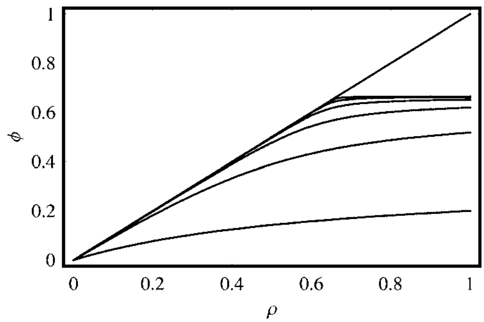

Fig. 2. The flow-density plots for $q = -1/2$ and $\delta\tau = 0.5, 0.5 \times 10^{-1}, 0.5 \times 10^{-2}, 0.5 \times 10^{-3}, 0.5 \times 10^{-4}, 0.5 \times 10^{-5}, 0$, going in bottom-up direction respectively.

Of course the above qualitative arguments being far from rigorous can only serve as an illustration. A specific distribution of the occupation numbers of MP sites can be obtained with the help of the canonical partition function formalism, which is described in details in the next section. In present section we analyse the behaviour of generating function $\Lambda_0(\gamma)$ of the cumulants of the integrated particle current Y_t that can be extracted form the solution of the Bethe ansatz equations discussed above.

To this end, we note that the universal scaling form (17–19) of $\Lambda_0(\gamma)$ holds in all the phase space, provided that the parameters ϕ, k_1, k_2 are finite. What depends on the parameters of the problem is the typical scales encoded in ϕ, k_1, k_2, which characterize the average and the fluctuations of the integrated particle current Y_t.

Before going to the analysis let us look how the results are related to those for the ASAP. To this end, we recall that in order to transform the ZRP to the ASAP we consider comoving reference frame, which takes one step forward at each time step. The distances, which particles travel in these reference frames, are related to each other as follows

$$Y_t^{ZRP} = Mt - Y_t^{ASAP}. \tag{83}$$

Hence, the largest eigenvalues of the equations for the generating functions $F_t\,(C)$ for the ASAP and ZRP are related as follows

$$\ln \Lambda_0\,(\gamma) = \gamma M - \delta\tau\lambda_0^{ASAP} + O(\delta\tau^2). \tag{84}$$

By λ_0^{ASAP} we mean the largest eigenvalue of the continuous time equation for the ASAP obtained in.[32] The factor $\delta\tau$ which comes with λ_0^{ASAP} reflects the time rescaling, in which one discrete time step is identified with the infinitesimal interval $\delta\tau$. The formula (84) suggests that the expansion of $\ln \Lambda_0\,(\gamma)$ in powers of $\delta\tau$ exists, which turns out to be true only in the subcritical region $\rho < \rho_c$. In what follows we solve the Eq. (80) in the limit $p \to 1$, i.e. $\delta\tau \to 0$. Once the value of x_c is obtained from this solution, it is to be directly substituted to the formulas (77–79) to obtain ϕ, k_1, k_2.

When $\delta\tau = 0$ the function $g_{q,\lambda}(x)$ can be summed up to the following simple form

$$g_{q,-1/q}(x) = \frac{x}{x-q}, \tag{85}$$

such that Eq. (80) can be easily solved

$$x_c = \frac{q\rho}{\rho - 1}. \tag{86}$$

Then Eqs. (77–79) yield: $\phi = \rho$ and $k_1 = 0$, i.e. $\ln \Lambda_0 = \gamma M$. This corresponds to synchronous jumps of all particles without any stochasticity. For small nonzero $\delta\tau$ we expect the correction of order of $\delta\tau$ to appear. This should be true at least if the argument of $g_{q,\lambda}(x)$ is away from any of its singularities, which turns out to be the case only in the subcritical region $\rho < \rho_c$. In general in this case one can represent $g_{q,\lambda}(x)$ as a Taylor series in powers of $\delta\tau$ with non-singular series coefficients. Then Eq. (80) can be solved perturbatively order by order in $\delta\tau$. To this end, we look for the solution in the form of the perturbative expansion in powers of $\delta\tau$

$$x_c = x_c^{(0)} + \delta\tau x_c^{(1)} + \cdots \tag{87}$$

The first order solution of Eq. (80) yields

$$x_c^{(1)} = \frac{(1-q)\rho}{(1-\rho)}\left(1 - \left(\sum_{s=1}^{\infty} \frac{q^s}{(1+(q^s-1)\rho)^2}\right)\right), \tag{88}$$

which results in the following values of ϕ, k_1, k_2:

$$\phi \simeq \rho + \delta\tau \left[\frac{(1-q)}{q}\left(\sum_{s=1}^{\infty} \frac{s}{1-q^s}\left(\frac{q\rho}{\rho-1}\right)^s\right)\right], \tag{89}$$

$$k_1 \simeq \delta\tau \frac{(q-1)}{\sqrt{8\pi}q\,((1-\rho)\rho)^{3/2}} \sum_{s=1}^{\infty} \frac{s^2(s-1+2\rho)}{1-q^s}\left(\frac{q\rho}{\rho-1}\right)^s, \tag{90}$$

$$k_2 \simeq \sqrt{2\pi(1-\rho)\rho}. \tag{91}$$

Thus, the first order calculation allows taking into account the fluctuations of the particle current due to spontaneous MP site formation. The values obtained indeed reproduce $\lambda_0^{\mathrm{ASAP}}$ from.[32] Close to the critical point $x_c^{(1)}$ diverges,

$$x_c^{(1)} \sim (\rho_c - \rho)^{-2},$$

resulting in the divergent contributions to ϕ, k_1, k_2

$$\phi^{(1)} \sim (\rho_c - \rho)^{-2}, k_1^{(1)} \sim (\rho_c - \rho)^{-4}, k_2^{(1)} \sim (\rho_c - \rho)^{-3}.$$

The reason of these divergencies have a clear physical meaning. For example, the first order correction to the flow ϕ gives a fraction of particles, which stay in MP sites. Roughly speaking it is a product, $t_{MP} n_{MP} p_{MP}$, of the life time of an MP site t_{MP}, its average occupation number n_{MP} and the probability of appearance p_{MP}, the latter being of order of $\delta\tau$. As it follows from the above arguments based on the balance between the flows into and out of an MP site, the two former, t_{MP} and n_{MP}, should diverge as the density of particles approaches its critical value.

The singularities in the solution limit the applicability of the perturbative scheme used. It to be applicable the solution x_c should be far enough from the first singularity of $g_{q,\lambda}(x)$, at the positive half-axis, $x = 1$. Specifically, the parameter $(\delta\tau x_c^{(1)}/(\rho_c - \rho))$ should be small to ensure the existence of nonsingular expansion for $g_{q,\lambda}(x)$. This yields

$$\delta\tau(\rho_c - \rho)^{-3} \ll 1, \tag{92}$$

resulting in

$$1 - x_c \sim \rho_c - \rho \gg \delta\tau^{1/3}. \tag{93}$$

To proceed further, we note that the function $g_{q,\lambda}(x)$ monotonously grows along the positive half-axis from zero at the origin, $x = 0$, to infinity at the pole $x = 1$. As the function $g_{q,\lambda}(x)$ runs over all positive real values between these two points, including the value of ρ, the solution of Eq. (80), x_c, is always located at the segment $0 \leq x \leq 1$. According to the above perturbative solution almost all this segment is exhausted by the subcritical domain, Eq. (93), except the small vicinity of $x = 1$, i.e. $(1 - x_c) \lesssim \delta\tau^{1/3}$. Therefore, this vicinity is where x_c should be looked for at the critical point and above.

This remark can be used as a basis for another perturbative scheme. Let us suppose that

$$x_c = 1 - \Delta, \tag{94}$$

where Δ is a small parameter. The function $g_{q,\lambda}(x)$ can be represented in the following form

$$g_{q,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\lambda x q^k}{1 + q^k \lambda x} + \sum_{k=0}^{\infty} \frac{q^k x}{1 - q^k x}. \tag{95}$$

This is obtained by expanding the expression under the sum in Eq. (70) in powers of q^n and then changing the order of summations. In this form $g_{q,\lambda}(x)$ has a form of the sum of terms, each with one of the following simple poles

$$x'_k = q^{-k}, \qquad x''_k = -q^{-k}\lambda^{-1}, \qquad k = 0, 1, 2, \ldots$$

The pairs of the poles x'_k and x''_{k-1} merge when $\delta\tau$ tends to zero, so that corresponding terms having equal absolute values and opposite signs cancel each other. As a result, for finite Δ the only term of $g_{q,\lambda}(x_c)$, which survives in the limit $\delta\tau \to 0$, is the one with the pole x''_0 (see Eq. (85)). On the other hand the term under the first sum in Eq. (95) with the pole $x'_0 = 1$ tends to infinity when Δ tends to zero. The compromise is to consider the two limits simultaneously. Then the divergent contribution from the pole x'_0 can be cancelled up to a finite constant by the term with the pole $x''_1 = -(q\lambda)^{-1}$. The value of the resulting constant, which depends on the relation between $\delta\tau$ and Δ, should be chosen such as to satisfy Eq. (80). Substitution of Eq. (94) to Eq. (80) shows that the value of $g_{q,\lambda}(x_c)$ to be finite in the limit $\Delta \to 0$, the following ratio should be kept constant.

$$\delta\tau/\Delta^2 \to const \tag{96}$$

Using this fact we can look for the relation between $\delta\tau$ and Δ in form of the expansion in Δ

$$\delta\tau = (1 - 1/q)^{-1}\left(\alpha\Delta^2 + \beta\Delta^3 + O(\Delta^4)\right). \tag{97}$$

The coefficient $(1 - 1/q)^{-1}$ is for the further convenience. Solving the Eq. (80) order by order in powers of Δ we fix α and β,

$$\alpha = \rho - \rho_c \tag{98}$$

$$\beta = \alpha(1 + \alpha) - q\,(1 - q)^{-2}. \tag{99}$$

Inverting the relation (97) we express x_c in terms of $\delta\tau$. Finally we substitute x_c into Eqs. (77–79). The cases $\alpha = 0$ and $\alpha > 0$ should be considered separately.

– In the case $\alpha > 0$, which corresponds to the density above critical, $\rho > \rho_c$, we obtain in the leading order

$$\phi \simeq \rho_c - \delta\tau^{1/2}\left[\frac{\rho_c^2(1-\rho_c)}{\rho-\rho_c}\right]^{1/2}, \tag{100}$$

$$k_1 \simeq \frac{3}{8\sqrt{\pi}}\rho_c\,(1-\rho_c)^{3/4}\,\frac{\delta\tau^{1/4}}{(\rho-\rho_c)^{7/4}}, \tag{101}$$

$$k_2 \simeq 2\sqrt{\pi}\,(1-\rho_c)^{1/4}\,\frac{(\rho-\rho_c)^{3/4}}{\delta\tau^{1/4}}. \tag{102}$$

As follows form these formulas the flow of particles ϕ in the thermodynamic limit is equal to the critical density ρ_c as expected. It is possible also to obtain $1/N$ correction to the thermodynamic value of the particle flow ϕ, which was shown to be universal being proportional to the nonlinear coefficient λ of the KPZ equation.[36]

$$\phi_N - \phi \simeq \frac{1}{N}\frac{3}{4}\frac{\rho_c\,(1-\rho_c)}{(\rho-\rho_c)} \tag{103}$$

The value of the correction diverges in the critical point. Surprisingly its value does not depend on $\delta\tau$. Higher cumulants, responsible for the fluctuations around the average flow grow when $\delta\tau$ tends to zero,

$$\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t} \sim N^{3(n-1)/2}\frac{(\rho-\rho_c)^{(3n-7)/4}}{\delta\tau^{(n-1)/4}}. \tag{104}$$

– *In the critical point* $\alpha = 0$, *i.e.* $\rho = \rho_c$, we obtain,

$$\phi_c \simeq \rho_c - \delta\tau^{1/3}\left[\rho_c^2\,(1-\rho_c)\right]^{1/3}, \tag{105}$$

$$k_1 \simeq \left(\frac{\rho_c}{\delta\tau}\right)^{1/3}\left[54\pi\,(1-\rho_c)\,\rho_c\right]^{-1/2}, \tag{106}$$

$$k_2 \simeq \sqrt{6\pi\,(1-\rho_c)\,\rho_c}. \tag{107}$$

The $1/N$ correction to the average flow looks as follows

$$\phi_N - \phi \simeq \frac{1}{N}\frac{1}{3}\left(\frac{\rho_c}{\delta\tau}\right)^{1/3}, \tag{108}$$

growing with the decrease of $\delta\tau$. The other cumulants have similar $\delta\tau$ behavior,

$$\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t} \sim N^{3(n-1)/2}\delta\tau^{-1/3}. \tag{109}$$

As follows from the results obtained, though the average flow saturates to a constant value at the critical point, its fluctuations grow when $\delta\tau$ goes to zero.

The generating function of cumulants $\Lambda_0(\gamma)$ can be related to the large deviation function for the particle current.[13] As a result the probability distribution of Y_t/t reads as follows

$$P\left(\frac{Y_t}{t}=y\right) = \exp\left[-\frac{t}{t_c}H\left(\frac{y-N\phi}{\mathcal{G}}\right)\right], \tag{110}$$

where the universal function $H(x)$ is given by the following parametric expression

$$H(x) = \beta G'(\beta) - G(\beta) \tag{111}$$

$$x = G'(\beta). \tag{112}$$

with two model dependent constants

$$t_c = N^{3/2}k_1^{-1}$$

$$\mathcal{G} = k_1 k_2.$$

The function $H(x)$ behaves as $(x-1)^2$ for $x \ll 1$ and as $x^{5/2}$ and $x^{3/2}$ as x goes to plus and minus infinity respectively. One can see from Eq. (110) that the parameter $\mathcal{G}$ plays the role of the characteristic scale at which the current fluctuations become non-Gaussian. This takes place when the argument of $H(x)$ in Eq. (110) becomes of order of 1. Note that $\mathcal{G}$ coincides with $1/N$ correction to the average flow. This is a reflection of the fact that the correlations between particle jumps, which cause deviations from the Gaussian behaviour, owe to the finiteness of the system and to the periodic boundary conditions. The parameter t_c is the characteristic time in which such fluctuations become unlikely. The characteristic time t_c being of order of $N^{3/2}$ signifies the KPZ behaviour characterized by the dynamical exponent $z = 3/2$. In fact, for the totally[9] and the partially[12] ASEP the inverse time t_c^{-1} coincides with the next to the largest eigenvalue of the master equation up to a constant of order of 1. We expect this to hold also in our case. Below the critical point $\rho < \rho_c$ we have $t_c \sim N^{3/2}\delta\tau^{-1}$ and $\mathcal{G} \sim \delta\tau$. Above the critical point, $\rho > \rho_c$, the characteristic time scales as $t_c \sim N^{3/2}\delta\tau^{-1/4}$ growing with the decay of $\delta\tau$ and the fluctuation scale is finite $\mathcal{G} \sim (\rho - \rho_c)^{-1}$ growing as the density approaches the critical point. Exactly at the critical point we obtain $\mathcal{G} \sim \delta\tau^{-1/3}$ and $t_c \sim \delta\tau^{1/3}N^{3/2}$. One can see that below the critical point even very small fluctuations are non-Gaussian. Above the critical point the range of Gaussian fluctuations is finite. At the critical point the fluctuations remain Gaussian in a very wide range. On the other hand the time of decay of the fluctuations at the critical point is much smaller than below and above.

6. CANONICAL PARTITION FUNCTION FORMALISM FOR THE STATIONARY STATE

In this section we show that, the partition function formalism,[7,38] allows a calculation of some physical quantities yielding the same results as obtained above. Particularly the results obtained in the saddle point approximation are equivalent to the results obtained from the above solution of the Bethe equations in the thermodynamic limit. We also analyze the change of the occupation number distribution under the phase transition in the limit $\delta\tau \to 0$.

6.1. Canonical Partition Function and Stationary Correlations

The partition function of the ZRP is defined as the normalization constant of the stationary distribution (5)

$$Z(N, M) = \sum_{\{n_i\}} \prod_{i=1}^{N} f(n_i)\, \delta(n_1 + \cdots + n_N), \tag{113}$$

where $f(n)$ is the one-site weight defined in Eq. (6). The sum can be represented in the form of the contour integral,

$$Z(N,M) = \oint \frac{(F(z))^N}{z^{M+1}} \frac{dz}{2\pi i}, \tag{114}$$

where $F(z)$ is the generating function of the one-site weights,

$$F(z) = \sum_{n=0}^{\infty} z^n f(n). \tag{115}$$

Once the partition function is known, it can be used to obtain stationary correlation functions. For example the probability $P(n)$ for a site to hold n particles is as follows,

$$P(n) = f(n) \frac{Z(N-1, M-n)}{Z(N,M)}. \tag{116}$$

Another tool characterizing the occupation number distribution is the generating function of its moments $\langle e^{\gamma n} \rangle$, which can be also represented in the form of contour integral

$$\begin{aligned} \langle e^{\gamma n} \rangle &= \sum_{k=0}^{\infty} \frac{\gamma^k \langle n^k \rangle}{k!} \\ &= \frac{1}{Z(N,M)} \oint \frac{(F(z))^{N-1} F(ze^{\gamma})}{z^{M+1}} \frac{dz}{2\pi i}. \end{aligned} \tag{117}$$

Another quantity of interest is the probability $\mathcal{P}(n)$ for n particles to hop simultaneously. The integral representation of the corresponding generating function is

$$\Psi(x) \equiv \sum_{n=0}^{M} x^n \mathcal{P}(n) = \frac{1}{Z(N,M)} \oint \frac{[\Phi(x,z)]^N}{z^{M+1}} \frac{dz}{2\pi i}, \tag{118}$$

where

$$\Phi(x,z) = \sum_{n=0}^{\infty} f(n) z^n (xp(n) + (1 - p(n))).$$

This yields for example the following expression for the average flow of particles ϕ_N,

$$\phi_N = \frac{1}{N} \Psi'(1) = \frac{1}{Z(N,M)} \oint \frac{[F(z)]^N}{z^{M+1}} \frac{z}{1+z} \frac{dz}{2\pi i}. \tag{119}$$

The subscript N specifies that the expression is valid for an arbitrary finite lattice.

6.2. The Saddle Point Approximation

The integral in Eq. (114) can be evaluated in the saddle point approximation. The location of the saddle point z^* is defined by the following equation.

$$\rho = z^* \frac{F'(z^*)}{F(z^*)}. \tag{120}$$

This equation can be treated as an equation of state, which relates the density of particles ρ and the fugacity of a particle z^*. It is convenient to introduce the Helmholtz free energy

$$\begin{aligned} A(N, M) &= -\ln Z(N, M) \\ &\simeq -N \ln F(z^*) + M \ln z^*. \end{aligned} \tag{121}$$

Then one can define the chemical potential

$$\mu = \left. \frac{\partial A(N, M)}{\partial M} \right|_N = \ln z^* \tag{122}$$

and the analogue of pressure

$$\pi = - \left. \frac{\partial A(N, M)}{\partial N} \right|_M = \ln F(z^*). \tag{123}$$

The stationary correlation functions mentioned above can also be expressed in terms of the values defined.

– The occupation number probability distribution $P(n)$ for $n \ll N$

$$P(n) = f(n) e^{n\mu - \pi}. \tag{124}$$

We should note that when n is of order of N this equation loses the validity as one should take into account the shift of the saddle point in the integral for $Z(N-1, M-n)$. In other words, when n in given site is large it changes the density of particles in the other sites, which in its turn leads to the change of the fugacity of particles.

– The cumulants $\langle n^k \rangle_c$ of the occupation number n are given by the derivatives of $\log \langle e^{\gamma n} \rangle$, Eq. (117), which leads to a standard relation between fluctuations and pressure

$$\langle n^k \rangle_c = \frac{\partial^k \pi}{\partial \mu^k}. \tag{125}$$

– The average flow of particles ϕ is given by

$$\phi = \frac{e^\mu}{1 + e^\mu}. \tag{126}$$

The $1/N$ correction to the average flow can be given in terms of two first cumulants of the occupation number $\langle n^2\rangle_c$, $\langle n^3\rangle_c$,

$$\phi_N - \phi \simeq \frac{1}{2N}\left(\frac{\partial\phi}{\partial\mu}\frac{\langle n^3\rangle_c}{\langle n^2\rangle_c^2} - \frac{\partial^2\phi}{\partial\mu^2}\frac{1}{\langle n^2\rangle_c}\right), \tag{127}$$

which in its turn can be reduced to a very simple form

$$\phi_N - \phi = -\frac{1}{2N}\frac{d}{d\mu}\left(\frac{d\phi}{d\rho}\right), \tag{128}$$

where one should take into account the equation of state (120). The $1/N$ correction to the average flow has been claimed to be universal for the KPZ class, depending only on the parameters of the corresponding KPZ equation. Below we use it to reexpress the above parameters k_1,k_2 obtained from the solution of the dynamical problem in terms of the parameters of the stationary state only.

We should note that the criterion of validity of the saddle point approximation, i.e. smallness of the second term of the asymptotical expansion comparing to the first, yields the following range of parameters:

$$\frac{\langle n^3\rangle_c^2}{\langle n^2\rangle_c^3} \ll \sqrt{N}. \tag{129}$$

Let us turn to the examples.

– *For the case* $q = 0$, the series $F(z)$ expressed in terms of the parameter $\lambda = p/(1-p)$ can be summed up to the form

$$F(z) = \frac{1}{1+\lambda} + \frac{z}{\lambda - z}. \tag{130}$$

In this case the integral in (114) can be evaluated explicitly resulting in

$$Z(N,M) = \lambda^{-M}(1+\lambda)^{-N}\frac{\Gamma(L)}{\Gamma(N)\Gamma(M+1)}\,{}_2F_1\left(\begin{matrix}-M,-N\\1-L\end{matrix};-\lambda\right). \tag{131}$$

Using then the formula (119) we arrive at the expression of the particle flow ϕ given in (65).

– *In the case of arbitrary* q, $|q| < 1$, due to specific form of the one site weights $f(n)$, the function $F(z)$ has the structure of well known q-series,

$$F(z) = (1-p)\sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^n\frac{(-\lambda;q)_n}{(q;q)_n}, \tag{132}$$

where

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \tag{133}$$

is a notation for shifted q-factorial. The series (132) can be summed up to the infinite product using the q-binomial theorem,[(31)] which is the q-analog of the Taylor series of a power law function,

$$F(z) = (1-p)\frac{(-z;q)_\infty}{(z/\lambda;q)_\infty}. \tag{134}$$

This formula (134) turns out to be extremely useful as it allows one to write the equation for the saddle point (120) in the following simple form.

$$\rho = \sum_{k=0}^{\infty} \frac{z^* q^k}{1+q^k z^*} + \sum_{k=0}^{\infty} \frac{z^* q^k/\lambda}{1-q^k z^*/\lambda} \tag{135}$$

Note that if we define

$$x_c = z^*/\lambda, \tag{136}$$

then the r.h.s of Eq. (135) coincides with $g_{q,\lambda}(x_c)$ in the form (95). Thus, the equation for the saddle point is nothing but the conical point equation, Eq. (80), appeared in the thermodynamic solution of the Bethe ansatz equations, while the above parameter x_c is proportional to the fugacity of a particle z^*.

Using the equality

$$g_{q,\lambda}(z/\lambda) = z\frac{F'(z)}{F(z)}$$

and Eq. (125) one can obtain the explicit form of the first two cumulants of the occupation number

$$\langle n^2\rangle_c = \frac{z^*}{\lambda} g'_{q,\lambda}(z^*/\lambda), \tag{137}$$

$$\langle n^3\rangle_c = \frac{z^*}{\lambda}\left(\frac{g'_{q,\lambda}(z^*/\lambda)}{\lambda} + \frac{z^* g''_{q,\lambda}(z^*/\lambda)}{\lambda^2}\right). \tag{138}$$

Substituting them into Eqs. (126,127) one can make sure that the expression for the particle flow coincides with that obtained from the Bethe ansatz solution.

It is remarkable to note that the variance of the occupation number, $\langle n^2\rangle_c$, can be directly related to the parameter k_2,

$$k_2 = \sqrt{2\pi\langle n^2\rangle_c} = \sqrt{2\pi\frac{\partial\rho}{\partial\mu}}. \tag{139}$$

At the same time the $1/N$ correction to the average flow, Eq. (127), is given by the product $k_1 k_2$ such that

$$k_1 k_2 = \frac{1}{2N}\left(\frac{\partial\phi}{\partial\mu}\frac{\langle n^3\rangle_c}{\langle n^2\rangle_c^2} - \frac{\partial^2\phi}{\partial\mu^2}\frac{1}{\langle n^2\rangle_c}\right)$$

$$= -\frac{1}{2}\frac{d}{d\mu}\left(\frac{d\phi}{d\rho}\right). \tag{140}$$

Thus, we have expressed the parameters characterizing the fluctuations of the particle current in terms of the parameters of the stationary state, which characterize the fluctuations of occupation numbers. As the obtaining of the latter does not requires an integrability, we expect this relation to hold for the general discrete time ZRP belonging to the KPZ class.

Another interesting relation can be found. Let us consider the free energy per site, $a(\rho, z)$ as a function of the density ρ and the fugacity z that formally can take on arbitrary complex values

$$a(\rho, z) = \lim_{N\to\infty}\frac{A(M, N)}{N} = -\ln F(z) + \rho \ln z.$$

Then the following relation holds

$$\frac{\partial a(\rho, z)}{\partial z} = \frac{1}{2\pi i}R_0(z),$$

where $R_0(z)$ is the density of the Bethe roots in the thermodynamic limit, Eq. (69). Thus the difference $|a(\rho, z_1) - a(\rho, z_2)|$ with the values of z_1, z_2 taken at the contour of the Bethe roots gives the fraction of the roots in the segment between $z_1$and z_2. We should also note that the equation of state, which in terms of $a(\rho, z)$ looks as follows

$$\partial a(\rho, z)/\partial z = 0,$$

is equivalent to the conical point equation Eq. (71).

6.3. The Occupation Number Distribution in the Deterministic Limit

As we noted above, the saddle point equation, Eq. (135), for the integral that represents the partition function $Z(M, N)$, Eq. (114), coincides with Eq. (80) appeared in the thermodynamic solution of the Bethe equations. Therefore, we can directly use the results of the solution of this equation obtained in the Sec. 5 to obtain the occupation number distribution in the limit $\delta\tau \to 0$.

– In the domain $\rho < \rho_c$ and $n \ll N$ we have

$$P(0) = 1 - \rho - O(\delta\tau),\ P(1) = \rho - O(\delta\tau) \tag{141}$$

$$P(n) = \delta\tau\frac{(1-\rho)(1-\rho_c)^{-2}}{(1-q^n)\left(1-q^{n-1}\right)}\left[\frac{(1-\rho_c)}{(1-\rho)}\frac{\rho}{\rho_c}\right]^n + O(\delta\tau^2), n \geq 2 \tag{142}$$

For $n \gg 1$ the latter formula yields the exponential decay.

$$P(n) \sim \delta\tau\frac{(1-\rho)}{(1-\rho_c)^2}\exp\left[-n\frac{\rho_c - \rho}{(1-\rho)\rho_c}\right]$$

When the density approaches the critical point from below, $(\rho_c - \rho) \ll 1$, the cut-off of this distribution diverges proportionally to $(\rho_c - \rho)^{-1}$. Note that strictly below ρ_c the cut-off is always finite, being independent of $\delta\tau$, while $P(n)$ itself is of order of $\delta\tau$.

– *In the domain $\rho > \rho_c$ and $n \ll N$ we have*

$$P(0) = 1 - \rho_c - O(\delta\tau^{1/2}),\; P(1) = \rho_c - O(\delta\tau^{1/2}) \tag{143}$$

and for $n \geq 2$

$$P(n) = \delta\tau \frac{(1-\rho_c)^{-1}}{(1-q^n)\left(1-q^{n-1}\right)}\left[1 - \sqrt{\frac{\delta\tau}{(1-\rho_c)(\rho-\rho_c)}}\right]^n + O(\delta\tau^{3/2}). \tag{144}$$

For large n the formula for $P(n)$ also yields the exponential decay

$$P(n) \sim \frac{\delta\tau}{(1-\rho_c)} \exp\left[-n\sqrt{\frac{\delta\tau}{(1-\rho_c)(\rho-\rho_c)}}\right]. \tag{145}$$

However, in this case the cut-off depends on $\delta\tau$ diverging as $\delta\tau^{-1/2}$ as $\delta\tau$ goes to zero.

– *When $\rho = \rho_c$ and $n \ll N$ we have*

$$P(0) = 1 - \rho_c - O(\delta\tau^{2/3}),\; P(1) = \rho_c - O(\delta\tau^{2/3}) \tag{146}$$

and for $n \geq 2$

$$P(n) \simeq \frac{\delta\tau(1-\rho_c)^{-1}}{(1-q^n)(1-q^{n-1})}\left[1 - \left(\frac{\delta\tau}{\rho_c(1-\rho_c)^2}\right)^{1/3}\right]^n + O\left(\delta\tau^{4/3}\right). \tag{147}$$

In this case the cut-off of $P(n)$ at large n is of order of $\delta\tau^{-1/3}$

$$P(n) \sim \frac{\delta\tau}{(1-\rho_c)} \exp\left[-n\left(\frac{\delta\tau}{\rho_c(1-\rho_c)^2}\right)^{1/3}\right]. \tag{148}$$

One can see from these formulas that below the critical point, the density of MP sites at the lattice vanishes proportionally to $\delta\tau$

$$\rho_{MP}(\rho < \rho_c) \equiv \sum_{n=2}^{M} P(n) \simeq \frac{\delta\tau}{\rho_c - \rho}\frac{\rho^2}{\rho_c}, \tag{149}$$

while their average occupation number $\langle n \rangle_{MP}$ is finite,

$$\langle n \rangle_{MP}(\rho < \rho_c) \equiv \frac{\sum_{n=2}^{M} nP(n)}{\sum_{n=2}^{M} P(n)} \simeq \frac{2\rho_c - \rho - \rho\rho_c}{\rho_c - \rho}, \tag{150}$$

both increasing as $(\rho_c - \rho)^{-1}$ as ρ approaches ρ_c.

Above ρ_c the density of MP sites vanishes as $\sqrt{\delta\tau}$,

$$\rho_{MP}(\rho > \rho_c) \simeq \sqrt{\frac{\delta\tau\,(\rho - \rho_c)}{(1-\rho_c)}}, \tag{151}$$

which is much slower than in the subcritical region. Their average occupation number diverges as $\delta\tau^{-1/2}$

$$\langle n \rangle_{MP}\,(\rho > \rho_c) \simeq \sqrt{\frac{(1-\rho_c)\,(\rho-\rho_c)}{\delta\tau}}. \tag{152}$$

Exactly at the critical point we have

$$\rho_{MP}(\rho = \rho_c) \simeq \frac{\delta\tau^{2/3}\rho_c^{1/3}}{(1-\rho_c)^{1/3}}, \tag{153}$$

$$\langle n \rangle_{MP}\,(\rho = \rho_c) \simeq \left(\frac{\rho_c}{\delta\tau}\right)^{1/3}(1-\rho_c)^{2/3}. \tag{154}$$

One can see that the fraction of particles in MP sites, i.e. $\rho_{MP}\,\langle n \rangle_{MP}$, above the critical point is exactly equal to $(\rho - \rho_c)$, which preserves the density of particles in MP sites equal to ρ_c.

The above formulas are derived under the suggestion $n \ll N$. For n of order of N one should take into account the shift of the saddle point in the numerator of Eq. (116). The criterion of the applicability of the saddle point method (129) suggests

$$\delta\tau^{1/2}N \gg 1 \tag{155}$$

for $\rho > \rho_c$ and

$$\delta\tau^{2/3}N \gg 1 \tag{156}$$

for $\rho = \rho_c$. Under these conditions the cutoffs of the distributions (146-148) are much smaller than N, and the probability of MP sites with $n \sim N$ is exponentially small in N. One can address the question what happens beyond the domains (155, 156). The extreme limiting case corresponds to the situation of the ASAP on a finite lattice, when $\delta\tau$ is so small that one can neglect any quantities of order of $\delta\tau^2$. In this case there is a tendency of creation of one MP site with the occupation number $N(\rho - \rho_c)$ and the life time that grows exponentially with the system size N.[35] The situation in the intermediate region should be addressed separately. Direct calculation of the occupation number distribution shows a peak in the middle, which grows as $\delta\tau$ approaches zero (see Fig (3)). Analysis of the finite lattice behaviour is beyond the goals of present article.

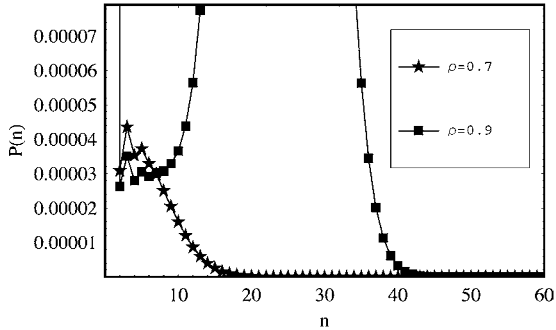

Fig. 3. The probability distributions $P(n)$ of the occupation number n for two values of density and $q = 0.5, \delta\tau = 10^{-5}, N = 100$.

7. SUMMARY AND DISCUSSION

We have presented the Bethe ansatz solution of the discrete time ZRP and ASEP with fully parallel update. We found the eigenfunctions of the equation for configuration-dependent generating function of the distance travelled by particles. The eigenfunctions of this equations were looked for in the form of the Bethe function weighted with the weights of stationary configurations. We started with the ZRP with arbitrary probabilities of the hopping of a particle out of a site, which depend on the occupation number of the site of departure. We inquired which restriction on the probabilities are imposed by the condition of the Bethe ansatz solvability. As a result we obtained the two-parametric family of probabilities, the two parameters being p, the hopping probability of a single particle, and q, the parameter responsible for the dependence of probability on the occupation number of a site. The model turns out to be very general including as particular cases q-boson asymmetric diffusion model, phase model, drop-push model and the ASAP. By simple arguments the ZRP can be also related to the ASEP-like system obeying exclusion interaction. In such system the hopping probabilities depend on the length of the queue of particles next to a hopping particle. The particular case of the model is the Nagel-Schreckenberg traffic model with $v_{\max} = 1$. We obtained the Bethe equations for both cases.

The further analysis was devoted to the calculation of generating function of cumulants of the distance travelled by particles. In the long time limit it is given by the maximal eigenvalue of the equation discussed. We obtained the largest eigenvalue in all the phase space of the model. First, the case $q = 0$ corresponding to the Nagel-Schreckenberg traffic model was considered, which due to special factorized form of the Bethe equations can be treated exactly for an arbitrary lattice size. The result is consistent with that for the continuous-time ASEP and the stationary solution of the Nagel-Schreckenberg traffic model. For the general

q we obtained the maximal eigenvalue in the scaling limit. It has the universal form specific for the KPZ universality class. Again, the model dependent constants obtained reproduce correctly all the known particular cases.

We carried out detailed analysis of the limiting case $p \to 1$. The phase transition which takes place in this limit was studied. We shown that below the critical density, the flow of particles consists of almost deterministic synchronous jumps of all particles. Above the critical density the new phase appears. The finite fraction of all particles gets stuck immobile at the vanishing fraction of sites. The fluctuations of the particle current become singular, nonanalytic in $(1-p)$.

In terms of the associated ASEP the transition studied is the analytic continuation of well-known jamming transition in the Nagel-Schreckenberg model with $\mathrm{v}_{\max} = 1$, which corresponds to a particular case of our model. The value of the critical density in that case, $\rho_c = 1/2$ (or $\rho_c = 1$ for associated ZRP), follows from the particle hole symmetry. Switching on the interaction between particles breaks this symmetry and as a result decreases ρ_c.

From the point of view of the hydrodynamics of the particle flow the reason of the jamming is the vanishing of the velocity of kinematic waves $v_{kin} = \partial\phi/\partial\rho$, which are responsible for the relaxation of inhomogeneities. As was argued in,[37] the situation $v_{kin} = 0$ leads to the appearance of shocks in the ASEP interpretation or the MP sites with large occupation numbers in ZRP. When $\rho > \rho_c$ and $p \to 1$, v_{kin} behaves as follows

$$v_{kin} = \partial\phi/\partial\rho = \frac{\sqrt{1-p}}{2}\left[\frac{\rho_c^2\,(1-\rho_c)}{(\rho-\rho_c)^3}\right]^{1/2},$$

while in the close vicinity of the critical point, $|\rho - \rho_c| \sim (1-p)^{1/3}$, the velocity of kinematic waves vanishes as $(1-p)^{1/3}$. Similar transition also was considered in the bus route model.[29] In that case however the rates $u(n)$ of hopping of a particle out of the site occupied by n particles depend on the external parameter λ, such that the limits $\lambda \to 0$ and $n \to \infty$ are not commutative. In our case the transition owes to taking a limit in only one hopping probability $p(1)$ and to the discrete parallel update.

We also obtained the canonical partition function of the stationary state of the discrete time ZRP. It is expressed in terms of the q-exponential series. We apply so called q-binomial theorem to evaluate the partition function in the saddle point approximation. Using the partition function formalism we obtained the formulas for physical quantities characterizing the stationary state of the model such as the occupation number distribution and the average flow of particles, the latter confirming the Bethe ansatz result. We observed several curious facts, which reveal intimate relation between the saddle point approximation applied for the partition function formalism and the thermodynamic limit of the Bethe ansatz solution. We found that two constants which define the behaviour of the large deviation function of the particle current can be expressed in terms of the cumulants of the occupation number of a site and the fugacity of particles. Furthermore, the analogue of the Helmholtz free energy considered as a function of arbitrary complex fugacity plays the role of a "counter" of the Bethe roots. The equation of state, which relates the density and the fugacity, coincides with the conical point equation, (71), which fixes the position of the Bethe roots contour. All the correspondences found seem not accidental. It was claimed in[34] that the large deviation function for the particle current should be universal for all models belonging to the KPZ universality class. On the other hand, once the density of the Bethe roots is obtained, the derivation of the nonuniversal parameters is straightforward. It would be natural to expect that the derivative of the free energy would serve as a generalization of the density of Bethe roots for the cases where the Bethe ansatz is unapplicable. Currently we do not have any explanation of the relations found. However, if they do exist, clarifying of their internal structure could give a way for study of a variety of systems which do not posses the Bethe ansatz integrability. We leave these questions for further work.

Another possibility of continuation of current study is as follows. In the particular case, $q = 0$, of our model, which corresponds to the Nagel-Schreckenberg traffic model with $v_{\max} = 1$, the Bethe equations have special factorized form. Usually, the consequence of such factorizability is the possibility of time-dependent correlation functions in the determinant form. This program was realized before for the continuous time ASEP[15] and the ASEP with backward ordered update.[16] We expect that the parallel update case is also treatable.

ACKNOWLEDGMENTS

AMP is grateful to V.B. Priezzhev for stimulating discussion. The work was supported partially by FCT grant SFR/BPD/11636/2002, projects POCTI/MAT/46176/2002 and POCTI/FAT/46241/2002, grant of Russian Foundation for Basic Research No.03-01-00780 and DYSONET project.

REFERENCES

1. M. Kardar, G. Parisi and Y.-C. Zhang, Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56**:889–892 (1986).
2. M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Discrete stochastic models for traffic flow. *Phys. Rev. E* **51**:2939–2949 (1995).
3. P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality: An explanation of the 1/f noise. *Phys. Rev. Lett.* **59**:381 (1987); *Phys. Rev. A* **38**:364 (1988).
4. J. Krug, Boundary-induced phase transitions in driven diffusive systems. *Phys. Rev. Lett.* **67**:1882 (1991).
5. G. Schütz and E. Domany, Phase transitions in an exactly soluble one-dimensional exclusion process. *J. Stat. Phys.* (**1–2**):277–296 (1993).

6. B. Derrida, S. A. Janovsky J. L. Lebowitz and E. R. Speer, Exact solution of the totally asymmetric simple exclusion process—shock profiles. *J. Stat. Phys.* **73**(5–6):813–842 (1993).
7. M. R. Evans, Phase transitions in one-dimensional nonequilibrium systems. *Brazilian J. Phy.* **30**:42 (2000).
8. D. Dhar, An exactly solved model for interfacial growth. *Phase Trans.* **9**:51 (1987).
9. L. H. Gwa and H. Spohn, Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation. *Phys. Rev. A* **46**:844 (1992).
10. G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. Lebowitz (Academic Press, London, 2000), Vol. 19.
11. B. Derrida, An exactly soluble non-equilibrium system: the asymmetric simple exclusion model. *Phys. Rep.* **301**:65 (1998).
12. D. Kim, Bethe ansatz solution for crossover scaling functions of the asymmetric XYZ chain and the Kardar-Parisi-Zhang type growth model. *Phys. Rev. E* **52**:3512 (1995).
13. B. Derrida and J. L. Lebowitz, Exact Large Deviation Function in the Asymmetric Exclusion Process. *Phys. Rev. Lett.* **80**:209–213 (1998).
14. D.-S. Lee and D. Kim, Large deviation function of the partially asymmetric exclusion process. *Phys. Rev. E* **59**:6476 (1999).
15. G. M. Schütz, Exact Solution of the Master Equation for the Asymmetric exclusion process. *J. of Stat. Phys.* **88**:1/2 (1997).
16. V. B. Priezzhev, Exact nonstationary probabilities in the asymmetric exclusion process on a ring. *Phys. Rev. Lett.* **91**:050601 (2003).
17. T. Sasamoto and M. Wadati, One-dimensional asymmetric diffusion model without exclusion. *Phys. Rev. E* **58**:4181 (1998).
18. T. Sasamoto and M. Wadati, Exact results for one-dimensional totally asymmetric diffusion models. *J. Phys. A: Math. Gen.* **31**:6057 (1998).
19. G. M. Schütz, R. Ramaswamy and M. J. Barma, Pairwise balance and invariant measures for generalized exclusion processes. *Phys. A: Math. Gen.* **29**:837 (1996).
20. M. Alimohammadi, V. Karimipour and M. Khorrami, Exact solution of a one-parameter family of asymmetric exclusion processes. *Phys. Rev. E* **57**:6370 (1998).
21. M. Alimohammadi, V. Karimipour and M. Khorrami, A Two-Parametric Family of Asymmetric Exclusion Processes and Its Exact Solution. *J. Stat. Phys.* **97**:373 (1999).
22. V. B. Priezzhev, E. V. Ivashkevich, A. M. Povolotsky and C.-K. Hu, Exact phase diagram for an asymmetric avalanche process. *Phys. Rev. Lett.* **87**:084301 (2001).
23. G. Schütz, Generalized Bethe ansatz solution of a one-dimensional asymmetric exclusion process on a ring with blockage. *J. Stat. Phys.* **71**:471 (1993).
24. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* . (Academic Press, 1982).
25. S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).
26. F. Spitzer, Interaction of Markov processes. *Adv. in Math.* **5**:246 (1970).
27. A. M. Povolotsky, Bethe ansatz solution of zero-range process with nonuniform stationary state. *Phys. Rev. E* **69**:061109 (2004).
28. N. M. Bogoliubov, A. G. Izergin and N. A. Kitanin, *Nucl. Phys B* **516**:501 (1998).
29. O. J. O'Loan, M. R. Evans and M. E. Cates, Jamming transition in a homogeneous one-dimensional system: The bus route model. *Phys. Rev. E* **58**:1404–1418 (1998).
30. E. Brunet and B. Derrida, Probability distribution of the free energy of a directed polymer in a random medium. *Phys. Rev. E* **61**(6): 6789–6801 (2000).
31. G. E. Andrews, R. Askey and R. Roy, *Special Functions, Encyclopedia of Mathematics and Its Applicatons* (Cambridge University Press, Cambridge, 1999).
32. A. M. Povolotsky, V. B. Priezzhev and Chin-Kun Hu, The asymmetric avalanche process. *J. Stat. Phys.* **111**:1149 (2003).

33. D. J. Bukman and J. D. Shore, The conical point in the ferroelectric six-vertex model. *J. Stat. Phys.* **78**:1277 (1995).
34. B. Derrida and C. Appert, Universal large deviation function of the Kardar-Parisi-Zhang equation in one dimension. *J. Stat. Phys.* **94**:1 (1999)
35. A. M. Povolotsky, V. B. Priezzhev and C.-K. Hu, Transition from Kardar-Parisi-Zhang to tilted interface critical behavior in a solvable asymmetric avalanche model. *Phys. Rev. Lett.* **91**:255701 (2003).
36. J. Krug, Origins of scale invariance in growth processes. *Adv. Phys.* **46**:139 (1997).
37. G. Tripathy and M. Barma, Steady state and dynamics of driven diffusive systems with quenched disorder. *Phys. Rev. Lett.* **78**:3039–3042 (1997).
38. P. Bialas, Z. Burda and D. Johnston, Condensation in the Backgammon model. *Nucl. Phys. B* **493**:505–516 (1997).